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Notes on Various Binomial Transforms of Generalized Pell Matrix Sequence

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Abstract

The main target of this study is to apply the binomial transform to the generalized Pell sequence. We define the binomial, s-binomial, rising, and falling transforms for generalized Pell matrix sequence. We establish some algebraic properties such as the recurrent formulas, Binet formu- las, generating functions, sum formulas etc... for generalized Pell matrix sequence.

Keywords: Binet formula, Binomial transforms, Generating function, Matrix sequences, Pell numbers.

1 | Introduction

The matrix sequences created from special integer sequences are very interesting topics in number and matrix theory. The Pell sequence is formed by adding twice the previous term to the term before that. When this sequence and its generalized sequences are expressed in matrices, some properties of the sequences can be obtained using matrix theory. Some papers focused on generalized Pell matrix sequences and their properties can be seen in [1], [2]. These papers give an idea for discovering potential applications of these sequences. For instance, Binet formula or generating functions and various relations are used to find the general behavior of the sequences. The Pell numbers p_n are defined by the recurrence relation $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$, beginning with the values $p_0 = 0$, $p_1 = 1$ in [1,2] [3], [4]. The generalized Pell sequence, called (s, t)-Pell sequence depending on two real parameters s, t and $s^2 + t > 0$ is introduced as

$$p_n(s,t) = 2sp_n-1(s,t) + tp_n-2(s,t),\ p_0(s,t) = 0, p_1(s,t) = 1,$$

for $n \geq 2$. The Binet formula of the (s, t)-Pell sequence is $P_n(s,t) = \frac{x_1^n - x_2^n}{x_1 - x_2}$, where $x_1 = s + \sqrt{s^2 + t}$, $x_2 = s - \sqrt{s^2} + t$, where x_1 and x_2 are the roots of the characteristic equation of the recurrence formula of the (s, t)-Pell

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sequenc. The main target of the paper is to apply various binomial transforms to the (s, t)-Pell matrix sequence and find some relations and properties of the new binomial transform sequences. Prodinger investigsted binomial transform in [5]. Chen [6] found some properties about the binomial transform in. Falcon and Plaza [7] investigated the binomial transforms of k-Fibonacci sequence. The authors studied the binomial transforms of the k-Lucas sequence in [8]. Yilmaz and Taskara [9] studied binomial transforms of the Padovan and Perrin matrix sequences. The authors investigated different binomial transforms of k-Jacobsthal sequences in [10]. In [11], Binomial transform of quadrapell sequences and quadrapell matrix sequences are dealt with. Uygun [12] computed the binomial transforms of the generalized (s, t)-Jacobsthal matrix sequencein. Kaplan and Özkoç Öztürk [13] studied on the binomial transforms of the Horadam quaternion sequences. Kwon [14] gave the binomial transforms of the modified k-Fibonacci-like sequence in. Soykan [15–21] sudied binomial transforms of the generalized Tribonacci sequence, the generalized third order Pell sequence, the generalized fourth order Pell sequence, the generalized fifth order Pell sequence, the generalized Narayana sequence, the generalized Pentanacci sequence, the binomial transform of the generalized Jacobsthal-Padovan numbers.

2 | Binomial Transform of (s, t)-Pell Matrix Sequences

Definition 1. The (s, t)-Pell matrix sequence is defined as

$$P_{n+1}(s,t) = 2sP_n(s,t) + tP_{n-1}(s,t),$$
$$P_0(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_1(s,t) = \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}. \tag{1}$$

For $n \geq 1$, any positive integer, s, t real numbers such that $s^2 + t > 0$.

Definition 2. The binomial transform of (s, t)-Pell matrix sequence indicated as $\{B_n(s,t)\}_{n\in\mathbb{N}}$ is defined as

$$B_n(s,t) = \sum_{i=0}^{n} \binom{n}{i} P_i(s,t). \tag{2}$$

For any positive integer s, t.

Lemma 1. The binomial transform of (s, t)-Pell matrix sequence satisfies the relation.

$$B_{n+1}(s,t) = \sum_{i=0}^{n} \binom{n}{i} [P_i(s,t) + P_{i+1}(s,t)].$$

Proof: By the property $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ and $\binom{n}{n+1} = 0$ and *Eq. (2)*, we have

$$B_{n+1}(s,t) = \sum_{i=1}^{n+1} \binom{n+1}{i} P_i(s,t) = P_0(s,t) + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1}\right] P_i(s,t)$$

$$= P_0(s,t) + \sum_{i=1}^{n} \binom{n}{i} P_i(s,t) + \sum_{i=1}^{n+1} \binom{n}{i-1} P_i(s,t)$$

$$= \sum_{i=0}^{n} \binom{n}{i} P_i(s,t) + \sum_{i=0}^{n} \binom{n}{i} P_i + 1\,(s,t).$$

Theorem 1. The recurrence relation of the binomial transform of (s, t)-Pell matrix sequence is as follows:

$$B_{n+1}(s,t) = (2 + 2s)B_n(s,t) + (t - 2s - 1)B_{n-1}(s,t). \tag{3}$$

Proof: By *Lemma 1* and *Eq. (1)*, it is obtained that

$$B_{n+1}(s,t) = \sum_{i=0}^{n} \binom{n}{i} \left(P_i(s,t) + P_{i+1}(s,t)\right)$$

$$= P_0(s,t) + P_1(s,t) + \sum_{i=1}^{n} \binom{n}{i} \left(P_i(s,t) + P_{i+1}(s,t)\right)$$

$$= P_0(s,t) + P_1(s,t) + \sum_{i=1}^{n} \binom{n}{i} (P_i(s,t) + 2sP_i(s,t) + tP_{i-1}(s,t))$$

$$= P_0(s,t) + P_1(s,t) + (1+2s)\sum_{i=1}^{n} \binom{n}{i} P_i(s,t) + t\sum_{i=1}^{n} P_{i-1}(s,t).$$

Then, by the definition of the binomial transform of (s, t)-Pell matrix sequence, we get

$$B_{n+1}(s,t) = -2sP_0(s,t) + P_1(s,t) + (1+2s)B_n(s,t) + t\sum_{i=1}^{n} \binom{n}{i} P_{i-1}(s,t). \tag{4}$$

Let's substitute for n in place of n + 1 in the last *Equality (4)*.

$$B_n(s,t) = -2sP_0(s,t) + P_1(s,t) + (1+2s)B_{n-1}(s,t)$$

$$+t\sum_{i=1}^{n-1} \binom{n-1}{i} P_{i-1}(s,t)$$

$$= -2sP_0(s,t) + P_1(s,t) + 2sB_{n-1}(s,t)$$

$$+\sum_{i=1}^{n} \binom{n-1}{i-1} P_{i-1}(s,t) + t\sum_{i=1}^{n-1} \binom{n-1}{i} P_{i-1}(s,t).$$

By $\binom{n-1}{n} = 0$, it is obtained that

$$B_n(s,t) = -2sP_0(s,t) + P_1(s,t) + 2sB_{n-1}(s,t)$$

$$+\sum_{i=1}^{n} \left[t\binom{n-1}{i} + \binom{n-1}{i-1}\right] P_{i-1}(s,t)$$

$$= -2sP_0(s,t) + P_1(s,t) + 2sB_{n-1}(s,t)$$

$$+\sum_{i=1}^{n} \left[t\binom{n-1}{i} + \binom{n-1}{i-1} + t\binom{n-1}{i-1} - t\binom{n-1}{i-1}\right] P_{i-1}(s,t)$$

$$= -2sP_0(s,t) + P_1(s,t) + 2sB_{n-1}(s,t)$$

$$+\sum_{i=1}^{n} \left[(1-t)\binom{n-1}{i-1} + t\binom{n}{i}\right] P_{i-1}(s,t)$$

$$= -2sP_0(s,t) + P_1(s,t) + 2sB_{n-1}(s,t)$$

$$+t\sum_{i=1}^{n} \binom{n}{i} P_{i-1}(s,t) + (1-t)\sum_{i=0}^{n-1} \binom{n-1}{i} P_i(s,t).$$

$$B_n(s,t) = -2sP_0(s,t) + P_1(s,t) + (2s+1-t)B_{n-1}(s,t) + t\sum_{i=1}^{n}\binom{n}{i}P_i(s,t). \quad \textbf{(5)}$$

By substituting the *Eq. (4)* into *Eq. (5)*, and by some algebraic operations, the proof is completed as

$$B_{n+1}(s,t) = (2+2s)B_n(s,t) + (t-2s-1)B_{n-1}(s,t).$$

Theorem 2. The Binet formula of the binomial transform of (s,t)-Pell matrix sequence is computed as

$$B_n(s,t) = \frac{[(1-b_2)P_0(s,t) + P_1(s,t)]b_1^n - [(1-b_1)P_0(s,t) + P_1(s,t)]b_2^n}{b_1 - b_2},$$

where $b_1 = s + 1 + \sqrt{s^2+t}$, $b_2 = s + 1 - \sqrt{s^2+t}$.

Proof: The characteristic polynomial equation of the recurrence *Relation (3)* is $x^2 - (2+2s)x - (t-2s-1) = 0$, whose solutions are b_1 and b_2. Assume that $B_n(s,t) = c_1 b_1^n + c_2 b_2^n$. By definition, we find that $B_0(s,t) = P_0(s,t)$ and $B_1(s,t) = P_1(s,t)$. Let us substitute for $n = 0$ and $n = 1$ in this equality, then we deduce that $c_1 = \frac{P_0(a,t)+P_1(a,t)-P_0(a,t)b_2}{b_1-b_2}$, $c_2 = \frac{P_0(a,t)b_1-P_1(a,t)-P_0(a,t)}{b_1-b_2}$. After substituting the values of c_1, c_2, we get the result.

Theorem 3. The generating function of the binomial transform of (s,t)-Pell matrix sequence is obtained as

$$B_n(s,t,x) = B_n = \sum_{i=0}^{\infty} B_i(s,t)x^i = \frac{B_0(s,t) + x\big(B_1(s,t) - (2+2s)B_0(s,t)\big)}{1-(2+2s)x-(t-2s-1)x^2}.$$

Proof: The generating function is a power series whose coefficients are the binomial transform of the (s,t)-Pell matrix sequence centered at the origin. By multiplying $B_n(s,t,x)$ by $-(2+2s)x$ and $-(t-2s-1)x^2$, it is obtained that

$$\begin{aligned} -(2+2s)xB_n &= -(2+2s)xB_0(s,t) - (2+2s)x^2B_1(s,t) + \cdots \\ -(t-2s-1)x^2B_n &= -(t-2s-1)x^2B_0(s,t) - (t-2s-1)x^3B_1(s,t) + \cdots. \end{aligned}$$

By these equalities and the recurrence *Relation (3)*, it is computed that

$$\begin{aligned} &[1-(2+2s)x-(t-2s-1)x^2]B_n \\ &= B_0(s,t) + x(B_1(s,t) - (2+2s)B_0(s,t)) \\ &\quad + x^2(B_2(s,t) - (2+2s)B_1(s,t) - (t-2s-1)B_0(s,t)) + \cdots \\ &= B_0(s,t) + x[B_1(s,t) - (2+2s)B_0(s,t)]. \end{aligned}$$

The proof is completed.

Theorem 4. Let n be a positive integer. Then the sum of the binomial transform of (s,t)-Pell sequence is given as

$$\sum_{i=0}^{p-1} B_{mi+n}(s,t) = \frac{\begin{gathered} B_n(s,t) - B_{mp+n}(s,t) - (2s-t+1)^n B_{m-n}(s,t) \\ +(2s-t+1)^m B_{m(p-1)+n}(s,t) \end{gathered}}{1 - \left(b_1^m + b_2^{m^2}\right) + (2s-t+1)^m}.$$

Proof: By the Binet formula of the binomial transform of (s,t)-Pell matrix sequence and the geometric sum property, it is computed as

$$\sum_{i=0}^{p-1} B_{mi+n}(s,t) = \sum_{i=0}^{p-1} c_1 b_1^{mi+n} + c_2 b_2^{mi+n} = c_1 b_1^n \sum_{i=0}^{p-1} b_1^{mi} + c_2 b_2^n \sum_{i=0}^{p-1} b_2^{mi}$$
$$= c_1 b_1^n \left(\frac{1-b_1^{mp}}{1-b_1^{mp}}\right) + c_2 b_2^n \left(\frac{1-b_2^{mp}}{1-b_2^{m}}\right)$$
$$= \frac{(c_1 b_1^n + c_2 b_2^n) - c_1 b_2^m b_1^n - c_2 b_1^m b_2^n - c_1 b_1^{mp+n} - c_2 b_2^{mp+n} + c_2 b_2^{mp+n} b_1^{mn} + c_1 b_1^{mp+n} b_2^m}{1-(b_1^m + b_2^m) + (b_1 b_2)^m}$$
$$= \frac{(c_1 b_1^n + c_2 b_2^n) - \left(c_1 b_1^{mp+n} + c_2 b_2^{mp+n}\right) - (b_1 b_2)^n [c_2 b_2^{m-n} + c_1 b_1^{m-n}] - (b_1 b_2)^m \left[c_2 b_2^{m(p-1)+n} + c_1 b_1^{m(p-1)+n}\right]}{1-(b_1^m + b_2^m) + (b_1 b_2)^m}.$$

We know that $b_1 b_2 = 2s - t + 1$. And by the Binet formula of the sequence, the result is obtained.

3 | The s-Binomial Transform of the (s,t)-Pell Matrix Sequences

Definition 3. The s-binomial transform of the (s,t)-Pell matrix sequence $\{O_n(s,t)\}_{n\in N}$ is demonstrated by

$$O_n(s,t) = \sum_{i=0}^{n} \binom{n}{i} s^n P_i(s,t). \quad \textbf{(6)}$$

It is easily seen that $O_n(s,t) = s^n B_n(s,t)$.

Lemma 2. The following relation for the s-binomial transform of the (s,t)-Pell matrix sequence verifies.

$$O_{n+1}(s,t) = \sum_{i=0}^{n} \binom{n}{i} s^{n+1} [P_i(s,t) + P_{i+1}(s,t)].$$

Proof: By the properties $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}, \binom{n}{n+1} = 0$, we get

$$O_{n+1}(s,t) = \sum_{i=0}^{n+1} \binom{n+1}{i} s^{n+1} P_i(s,t)$$
$$= s^{n+1} P_0(s,t) + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1}\right] s^{n+1} P_i(s,t)$$
$$= P_0(s,t) + \sum_{i=1}^{n+1} \binom{n}{i} P_i(s,t) + \sum_{i=1}^{n+1} \binom{n}{i-1} P_i(s,t)$$
$$= \sum_{i=0}^{n} \binom{n}{i} s^{n+1} P_i(s,t) + \sum_{i=0}^{n} \binom{n}{i} s^{n+1} P_{i+1}(s,t).$$

Theorem 5. The recurrence formula of s-the binomial transform of the (s,t) Pell matrix sequence is demonstrated as

$$O_{n+1}(s,t) = s(2+2s) O_n(s,t) + s^2(t-2s-1) O_{n-1}(s,t). \quad \textbf{(7)}$$

Proof: The initial conditions are found by *Eq. (6)* as $O_0(s,t) = P_0(s,t)$ and $O_1(s,t) = sP_1(s,t)$. By *Definition 1*, we obtain

$$
\begin{aligned}
O_{n+1}(s,t) &= \sum_{i=0}^{n} \binom{n}{i} s^{n+1}\left(P_i(s,t) + P_{i+1}(s,t)\right) \\
&= s^{n+1}\left(P_0(s,t) + P_1(s,t)\right) \\
&\quad + \sum_{i=1}^{n} \binom{n}{i} s^{n+1}\left(P_i(s,t) + P_{i+1}(s,t)\right) \\
&= s^{n+1}\left(P_0(s,t) + P_1(s,t)\right) \\
&\quad + \sum_{i=1}^{n} \binom{n}{i} s^{n+1}\left(P_i(s,t) + 2sP_i(s,t) + tP_{i-1}(s,t)\right) \\
&= s^{n+1}\left(P_0(s,t) + P_1(s,t)\right) \\
&\quad + (1+2s)\sum_{i=1}^{n} \binom{n}{i} s^{n+1} P_i(s,t) + t\sum_{i=1}^{n} \binom{n}{i} s^{n+1} P_{i-1}(s,t).
\end{aligned}
$$

$$
\begin{aligned}
O_{n+1}(s,t) &= s^{n+1}(P_1(s,t) - 2sP_0(s,t)) + s(1+2s)O_n(s,t) \\
&\quad + t\sum_{i=1}^{n} \binom{n}{i} s^{n+1} P_{i-1}(s,t).
\end{aligned} \tag{8}
$$

If we replace n by $n+1$ in *Eq. (8)*, it is obtained that

$$
\begin{aligned}
O_n(s,t) &= s^n(P_1(s,t) - 2sP_0(s,t)) + s(1+2s)O_{n-1}(s,t) \\
&\quad + t\sum_{i=1}^{n-1} \binom{n-1}{i} s^n P_{i-1}(s,t) \\
&= s^n(P_1(s,t) - 2sP_0(s,t)) + 2s^2 O_{n-1}(s,t) + \sum_{i=0}^{n-1} \binom{n-1}{i} s^n P_{i-1}(s,t) \\
&\quad + t\sum_{i=1}^{n-1} \binom{n-1}{i} s^n P_{i-1}(s,t) \\
&= s^n(P_1(s,t) - 2sP_0(s,t)) + 2s^2 O_{n-1}(s,t) + \sum_{i=1}^{n} \binom{n-1}{i-1} s^n P_i(s,t) \\
&\quad + t\sum_{i=1}^{n-1} \binom{n-1}{i} s^n P_{i-1}(s,t).
\end{aligned}
$$

Then, we get

$$
O_n(s,t) = s^n(P_1(s,t) - 2sP_0(s,t)) + 2s^2 O_{n-1}(s,t)
$$

$$
+ \sum_{i=1}^{n} \left[t\binom{n-1}{i} + \binom{n-1}{i-1} \right] s^n P_{i-1}(s,t)
$$

$$
= s^n(P_1(s,t) - 2sP_0(s,t)) + 2s^2 O_{n-1}(s,t)
$$

$$
+ \sum_{i=1}^{n} \left[t\binom{n-1}{i} + \binom{n-1}{i-1} + t\binom{n-1}{i-1} - t\binom{n-1}{i-1} \right] s^n P_{i-1}(s,t)
$$

$$= s^n(P_1(s,t) - 2sP_0(s,t)) + 2s^2 O_{n-1}(s,t)$$

$$+\sum_{i=1}^{n}\left[(1-t)\binom{n-1}{i-1} + t\binom{n}{i}\right] s^n P_{i-1}(s,t)$$

$$= s^n(P_1(s,t) - 2sP_0(s,t)) + 2s^2 O_{n-1}(s,t)$$

$$+t\sum_{i=1}^{n}\binom{n}{i} s^n P_{i-1}(s,t) + (1-t)\sum_{i=0}^{n-1}\binom{n-1}{i} s^n P_i(s,t).$$

$$O_n(s,t) = s^n(P_1(s,t) - 2sP_0(s,t)) + 2s^2 O_{n-1}(s,t)$$

$$+t\sum_{i=1}^{n}\binom{n}{i} s^n P_{i-1}(s,t) + (1-t)sO_{n-1}(s,t). \quad \textbf{(9)}$$

By substituting the *Equality (8)* into *Equality (9)*, we get

$$\begin{aligned} sO_n(s,t) &= s^{n+1}\big(P_1(s,t) - 2sP_0(s,t)\big) + 2s^3 O_{n-1}(s,t) + O_{n+1}(s,t) \\ &\quad - s^{n+1}\big(P_1(s,t) - 2sP_0(s,t)\big) - (1+2s)sO_n(s,t) + (1-t)s^2 O_{n-1}(s,t) \\ &= (2s^3 - ts^2 + s^2)O_{n-1}(s,t) - (1+2s)O_n(s,t) + O_{n+1}(s,t). \end{aligned}$$

The proof is found after some simple calculations as

$$O_{n+1}(s,t) = s(2+2s)O_n(s,t) + s^2(t-2s-1)O_{n-1}(s,t).$$

Theorem 6. The Binet formula of the s-binomial transform of (s,t)-Pell matrix sequence is demonstrated by

$$O_n(s,t) = \frac{[(s-o_2)P_0(s,t) + sP_1(s,t)]o_1^n - [(s-o_1)P_0(s,t) + sP_1(s,t)]o_2^n}{o_1 - o_2},$$

where $o_1 = s(s+1) + s\sqrt{s^2+t}, o_2 = s(s+1) - s\sqrt{s^2+t}$.

Proof: The characteristic equation of the recurrence *Eq. (7)* is obtained as $x^2 - s(2+2s)x + s^2(2s+t-1) = 0$, whose roots are o_1 and o_2 where $o_1 = s(s+1) + s\sqrt{s^2+t}, o_2 = s(s+1) - s\sqrt{s^2+t}$. Assume that $O_n(s,t) = c_1 o_1^n + c_2 o_2^n$. By definition $O_0(s,t) = P_0(s,t)$ and $O_1(s,t) = sP_1(s,t)$. Let us substitute for $n = 0$ and $n = 1$ in the equality $O_n(s,t)$, then we deduce that $c_1 = \frac{(s-o_2)P_0(s,t)+sP_1(s,t)}{o_1-o_2}, c_2 = \frac{(s-o_1)P_0(s,t)+sP_1(s,t)}{O_1-o_2}$. We can easily see the result after substituting these values into $O_n(s,t)$.

Theorem 7. The generating function for the s-binomial transform of (s,t) Pell matrix sequence is obtained as

$$O_n(s,t,x) = O_n = \sum_{i=0}^{\infty} O_i(s,t)x^i = \frac{O_0(s,t) + x[O_1(s,t) - s(2+2s)O_0(s,t)]}{1 - s(2+2s)x + s^2(2s-t+1)x^2}.$$

Proof: By multiplication $O_n(s,t,x)$ by $-s(2+2s)x$ and $s^2(2s-t+1)x^2$, the following equlities are obtained

$$-s(2+2s)xO_n = -s(2+2s)xO_0(s,t) - s(2+2s)x^2 O_1(s,t) + \cdots$$

$$s^2(2s-t+1)x^2 O_n = s^2(2s-t+1)x^2 O_0(s,t) + s^2(2s-t+1)x^3 O_1(s,t) + \cdots.$$

From these equalities and the recurrence *Relation (7)*, the generating function is obtained.

Theorem 8. Let m, n be any positive integers. Then the sum of the s-binomial transform of (s,t)-Pell sequence is given as

$$\sum_{i=0}^{p-1} O_{mi+n}(s,t) = \frac{\begin{matrix} O_n(s,t) - O_{mp+n}(s,t) - [s^2(2s-t+1)]^n O_{m-n}(s,t) \\ +[s^2(2s-t+1)]^m O_{m(p-1)+n}(s,t) \end{matrix}}{1-(o_1^m + o_2^m) + [s^2(2s-t+1)]^m}.$$

Proof: By the Binet formula of the s-binomial transform of (s,t)-Pell matrix sequence and the geometric sum property, the result is computed by using the same method in *Theorem 4*.

4 | The Rising Binomial Transform of the (s,t) Pell Matrix Sequences

Definition 4. The rising binomial transform of the (s,t)-Pell matrix sequence $\{I_n(s,t)\}_{n\in N}$ is defined by the following formula

$$I_n(s,t) = \sum_{i=0}^{n} \binom{n}{i} s^i P_i(s,t). \tag{10}$$

Lemma 3 ([2]). The Binet formula of the (s,t)-Pell matrix sequence is

$$P_n(s,t) = Mx_1^n - Nx_2^n,$$

where $M = \frac{P_1 - x_2 P_0}{x_1 - x_2}$, $N = \frac{P_1 - x_1 P_0}{x_1 - x_2}$, $x_1 = s + \sqrt{s^2+t}$, $x_2 = s - \sqrt{s^2+t}$.

Theorem 9. The Binet formula for the rising binomial transform of the (s,t) Pell matrix sequence is

$$I_n(s,t) = M(sx_1+1)^n - N(sx_2+1)^n,$$

where $x_1 = s - \sqrt{s^2+t}$, $x_2 = s + \sqrt{s^2+t}$.

Proof: In *Eq. (14)*, we have

$$\begin{aligned} I_n(s,t) &= \sum_{i=0}^{n} \binom{n}{i} s^i P_i(s,t) = \sum_{i=0}^{n} \binom{n}{i} s^i \left(Mx_1^i - Nx_2^i\right) \\ &= M\sum_{i=0}^{n} \binom{n}{i} (sx_1)^i - N\sum_{i=0}^{n} \binom{n}{i} (sx_2)^i \\ &= M(sx_1+1)^n - N(sx_2+1)^n). \end{aligned}$$

Theorem 10. For $n \geqslant 1$, the rising binomial transform of the (s,t)-Pell matrix sequence is a recurrence sequence such that

$$I_{n+1}(s,t) = (2s^2+2)I_n(s,t) - (1 - s^2t + 2s^2)I_{n-1}(s,t).$$

with initial conditions $I_n(s,t) = P_0(s,t)$ and $I_1(s,t) = sP_1(s,t)$.

Proof: By Binet formula for the rising binomial transform of the (s,t)-Pell matrix sequence, we get

$$
\begin{aligned}
&(2s^2+2)I_n(s,t)-(1-s^2t+2s^2)I_{n-1}(s,t)\\
&=(2s^2+2)[M(sx_1+1)^n-N(sx_2+1)^n]\\
&\quad-(1-s^2t+2s^2)[M(sx_1+1)^{n-1}-N(sx_2+1)^{n-1}]\\
&=\begin{Bmatrix} M(sx_1+1)^{n-1}[(2s^2+2)(sx_1+1)-(1-s^2t+2s^2)] \\ -N(sx_2+1)^{n-1}[(2s^2+2)(sx_2+1)-(1-s^2t+2s^2)] \end{Bmatrix}\\
&=\begin{Bmatrix} M(sx_1+1)^{n-1}[s^2(2sx_1+t)+2sx_1+1] \\ -N(sx_2+1)^{n-1}[s^2(2sx_2+t)+2sx_2+1] \end{Bmatrix}\\
&=\begin{Bmatrix} M(sx_1+1)^{n-1}[(sx_1+1)^2] \\ -N(sx_2+1)^{n-1}[(sx_2+1)^2] \end{Bmatrix}\\
&=I_{n+1}(s,t).
\end{aligned}
$$

Theorem 11. The generating function of rising binomial transform of the (s,t) Pell matrix sequence is given as

$$I_n(s,t,x)=I_n=\frac{I_0(s,t)+x[I_1(s,t)-(2s^2+2)I_0(s,t)]}{1-(2s^2+2)x+(1-s^2t+2s^2)x^2}.$$

Proof: By following same procedure, we have

$$-(2s^2+2)xI_n=-(2s^2+2)xI_0(s,t)-(2s^2+2)x^2I_1(s,t)+\cdots$$

$$(1-s^2t+2s^2)x^2I_n=(1-s^2t+2s^2)x^2I_0(s,t)+(1-s^2t+2s^2)x^3I_1(s,t)+\cdots$$

$$[1-(2s^2+2)x+(1-s^2t+2s^2)x^2]I_n=I_0(s,t)-(2s^2+2)xI_0(s,t)+xI_1(s,t)+x^2(0).$$

By the above computations, we obtain the generating function.

5 | The Falling Binomial Transform of the (s,t) Pell Matrix Sequences

Definition 5. The falling binomial transform of the (s,t)-Pell matrix sequence $\{D_n(s,t)\}_{n\in N}$ is given by

$$D_n(s,t)=\sum_{i=0}^{n}\binom{n}{i}s^{n-i}P_i(s,t). \tag{11}$$

Lemma 4. The falling binomial transform of the (s,t)-Pell matrix sequence verifies the relation.

$$D_{n+1}(s,t)=\sum_{i=0}^{n}\binom{n}{i}s^{n-i}[sP_i(s,t)+P_{i+1}(s,t)],$$

with initial conditions $D_0(s,t)=P_0(s,t)$ and $D_1(s,t)=sP_0(s,t)+P_1(s,t)$.
Proof. In *Eq. (11)*, we take $n+1$ in place of n. After some computations, we obtain the following:

$$
\begin{aligned}
D_{n+1}(s,t)&=\sum_{i=0}^{n+1}\binom{n+1}{i}s^{n+1-i}P_i(s,t)\\
&=s^{n+1}P_0(s,t)+\sum_{i=1}^{n+1}\left[\binom{n}{i}+\binom{n}{i-1}\right]s^{n+1-i}P_i(s,t)\\
&=s^{n+1}P_0(s,t)+\sum_{i=1}^{n}s^{n+1-i}\binom{n}{i}P_i(s,t)+\sum_{i=0}^{n+1}s^{n-i}\binom{n}{i}P_{i+1}(s,t)\\
&=\sum_{i=0}^{n}s^{n+1-i}\binom{n}{i}P_i(s,t)+\sum_{i=0}^{n}s^{n-i}\binom{n}{i}P_{i+1}(s,t).
\end{aligned}
$$

Theorem 12. The following recurrence relation is verified by the falling binomial transform of the (s,t)-Pell matrix sequence.

$$D_{n+1}(s,t) = 4sD_n(s,t) + (3s^2 - t)D_{n-1}(s,t).$$

Proof: The initial conditions are found by *Definition 5* as $D_0(s,t) = P_0(s,t)$ and $D_1(s,t) = sP_0(s,t) + P_1(s,t)$. By *Lemma 4* and *Eq. (1)*, it is obtained that

$$\begin{aligned}
D_{n+1}(s,t) &= \sum_{i=0}^{n} \binom{n}{i} s^{n-i}[sP_i(s,t) + P_{i+1}(s,t)] \\
&= s^n[sP_0(s,t) + P_1(s,t)] + \sum_{i=1}^{n} \binom{n}{i} s^{n-i}[sP_i(s,t) + P_{i+1}(s,t)] \\
&= s^n[sP_0(s,t) + P_1(s,t)] + 3s\sum_{i=1}^{n} \binom{n}{i} s^{n-i}P_i(s,t) + t\sum_{i=1}^{n} \binom{n}{i} s^{n-i}P_{i-1}(s,t) \\
&= s^n[P_1(s,t) - 2sP_0(s,t)] + 3sD_n(s,t) + t\sum_{i=1}^{n} \binom{n}{i} s^{n-i}P_{i-1}(s,t).
\end{aligned} \tag{12}$$

Let's substitute n in place of $n+1$ in *Equality (12)*.

$$\begin{aligned}
D_n(s,t) =& s^{n-1}[P_1(s,t) - 2sP_0(s,t)] + 3sD_{n-1}(s,t) \\
&+ t\sum_{i=1}^{n-1} \binom{n-1}{i} s^{n-1-i}P_{i-1}(s,t) \\
=& s^{n-1}[P_1(s,t) - 2sP_0(s,t)] + 3s\sum_{i=0}^{n-1} \binom{n-1}{i} s^{n-1-i}P_i(s,t) \\
&+ t\sum_{i=1}^{n-1} \binom{n-1}{i} s^{n-1-i}P_{i-1}(s,t) \\
=& s^{n-1}[P_1(s,t) - 2sP_0(s,t)] + 3s\sum_{i=1}^{n-1} \binom{n-1}{i-1} s^{n-i}P_{i-1}(s,t) \\
&+ \frac{t}{s}\sum_{i=1}^{n-1} \binom{n-1}{i} s^{n-i}P_{i-1}(s,t).
\end{aligned}$$

We know that $\binom{n-1}{n} = 0$, hence we get

$$\begin{aligned}
D_n(s,t) &= s^{n-1}[P_1(s,t) - 2sP_0(s,t)] \\
&+ \sum_{i=1}^{n} \left[\frac{t}{s}\binom{n-1}{i} + 3s\binom{n-1}{i-1}\right] s^{n-i}P_{i-1}(s,t) \\
&= s^{n-1}[P_1(s,t) - 2sP_0(s,t)] \\
&+ \sum_{i=1}^{n} \left[\frac{t}{s}\binom{n-1}{i} + 3s\binom{n-1}{i-1} + \frac{t}{s}\binom{n-1}{i-1} - \frac{t}{s}\binom{n-1}{i-1}\right] s^{n-i}P_{i-1}(s,t)
\end{aligned} \tag{13}$$

$$= s^{n-1}[P_1(s,t) - 2sP_0(s,t)]$$

$$+\sum_{i=1}^{n}\left[\left(3s-\frac{t}{s}\right)\binom{n-1}{i-1}+\frac{t}{s}\binom{n}{i}\right]s^{n-i}P_{i-1}(s,t)$$

$$= s^{n-1}[P_1(s,t) - 2sP_0(s,t)]$$

$$+\frac{t}{s}\sum_{i=1}^{n}\binom{n}{i}s^{n-i}P_{i-1}(s,t)+\left(3s-\frac{t}{s}\right)\sum_{i=0}^{n-1}\binom{n-1}{i}s^{n-1-i}P_{i-1}(s,t) \tag{13}$$

$$= s^{n-1}[P_1(s,t) - 2sP_0(s,t)]$$

$$+\frac{t}{s}\sum_{i=1}^{n}\binom{n}{i}s^{n-1}P_{i-1}(s,t)+\left(3s-\frac{t}{s}\right)D_{n-1}(s,t).$$

By substituting *Eq. (12)* into *Eq. (13)* and by some simple calculations, the proof is completed as

$$D_{n+1}(s,t) = 4sD_n(s,t) + (3s^2 - t)D_{n-1}(s,t). \tag{14}$$

Theorem 13. The Binet formula for the falling binomial transform of (s,t)-Pell matrix sequence is calculated as

$$D_n(s,t) = \frac{[(s-d_2)P_0(s,t)+sP_1(s,t)]d_1^n - [(s-d_1)P_0(s,t)+sP_1(s,t)]d_2^n}{d_1-d_2},$$

where $d_1 = 2s + \sqrt{s^2} + t$, $d_2 = 2s - \sqrt{s^2} + t$.

Proof: The characteristic polynomial equation of recurrence *Eq. (14)* is $x^2 - 4sx + 3s^2 - t = 0$, whose solutions are $d_1 = 2s + \sqrt{s^2+t}$, $d_2 = 2s - \sqrt{s^2} + t$. Assume that $D_n(s,t) = c_1d_1^n + c_2d_2^n$. By definition $D_0(s,t) = P_0(s,t)$ and $D_1(s,t) = sP_1(s,t)$. Let us substitute for $n = 0$ and $n = 1$ in this equality, then we deduce that $c_1 = \frac{(s-d_2)P_0(s,t)+sP_1(s,t)}{d_1-d_2}$, $c_2 = -\frac{(s-d_1)P_0(s,t)+sP_1(s,t)}{d_1-d_2}$. After substituting the values of c_1 and c_2, we get the result.

Theorem 14. The generating function of the falling binomial transform of (s,t) Pell matrix sequence is as follows:

$$D_n(s,t,x) = \frac{D_0(s,t) + x[D_1(s,t) - 4sD_0(s,t)]}{1 - 4sx + (3s^2 - t)x^2}.$$

Proof: If we product the equality $D_n(s,t,x)$ by $-4sx$ and $(3s^2 - t)x^2$, it is obtained that

$$-4sxD_n(s,t,x) = -4sxD_0(s,t) - 4sx^2D_1(s,t) + \cdots$$
$$(3s^2 - t)x^2D_n(s,t,x) = (3s^2 - t)x^2D_0(s,t) + (3s^2 - t)x^3D_1(s,t) + \cdots$$

From the above equalities and *Theorem 12*, we get the desired result.

$$D_n(s,t,x) = \frac{D_0(s,t) + x[D_1(s,t) - 4sD_0(s,t)]}{1 - 4sx + (3s^2 - t)x^2}.$$

Theorem 15. Assume that n is a positive integer. Then the sum of the s binomial transform of (s,t)-Pell matrix sequence is given as

$$\sum_{i=0}^{p-1} D_{mi+n}(s,t) = \frac{\begin{array}{c} D_n(s,t) - (3s^2 - t)^n D_{m-n} - D_{mp+n}(s,t) \\ +(3s^2 - t)^m D_{m(p-1)+n}(s,t) \end{array}}{1 - (d_1^m + d_2^m) + (3s^2 - t)^m}.$$

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It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

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