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The Second-Order Constant-Coefficient Linear Homogeneous Weak Fuzzy Complex Differential Equations (WFC-DEs) and Initial Value Problems (WFC-IVPs)

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Abstract

This work aims to introduce the concept of the second-order Weak Fuzzy Complex-Differential Equations (WFC-DEs) for the first time. A special isomorphic transformation function could write a WFC-DE as two related Differential Equations (DEs) of the second order concerning their real variables. We study the second-order constant-coefficient linear homogeneous DE in a Weak Fuzzy Complex (WFC) variable with WFC constant coefficients. Also, we study the simple form of this type of WFC-DE with real constant coefficients. Thus, to find the general solution, we use the characteristic equation of the second-order DE. Also, we get a particular solution for the second-order constant-coefficient linear homogeneous Weak Fuzzy Complex-Initial Value Problem (WFC-IVP). To enhance understanding, we provide illustrative examples for each problem discussed.

Keywords: Weak fuzzy complex numbers, Quadratic weak fuzzy complex equation, Weak fuzzy complex functions, Weak fuzzy complex differential equation, Initial value problem.

1 | Introduction

Throughout history, the expansion of number sets has frequently arisen from the necessity to address particular mathematical challenges. Scholars have shown considerable interest in extending the set of real

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numbers R, with these newly formulated sets being characterized as two-dimensional generalizations of the real number system such as:

Complex numbers: $\{u + v\,i;\ u, v \in R;\ i^2 = -1\}$, dual numbers: $\{u + vi;\ u, v \in R;\ i^2 = 0\}$, Neutrosophic numbers: $\{u + vi;\ u, v \in R;\ i^2 = i\}$, and split-complex numbers: $\{u + v\,i;\ u, v \in R;\ i^2 = 1\}$.

Fuzzy numbers emerged as an extension of fuzzy set theory, which Lotfi Zadeh [1], [2] introduced in 1965. Fuzzy numbers represent uncertain numerical values, while traditional real numbers have a precise value. Nowadays, the theory of fuzzy sets has found applications across diverse domains, including artificial intelligence, biology, control systems, decision-making [3], [4], internet [5], information technology [4], electric vehicle [6], economics-industry [7], [8], and portfolio investment [9].

In 2023, the WFC set $F_J = \{u + v\,J;\ u, v \in R; J^2 = t \in]0,1[\}$ was defined in [10] as a new generalization of classical real numbers set R with fuzzy operators $J \notin R$. Also, they solved linear and quadratic WFC equations [10]. Later, they introduced the WFC vector space [11], the inner products defined over it [12]. Then, Alhasan et al. [13] studied the algebraic properties of WFC matrices and linear systems by applying these matrices.

Furthermore, the linear [14] and the nonlinear [15], [16] Diophantine equation in two WFC variables was studied with the concepts of the WFC and anti-WFC Pythagoras triples and Pythagoras quadruples. However, in [17], main notions were introduced in number theory, like division, ordering, and units in the WFC integers set. Alhasan et al. [18] showed that 'A-Curves' display the geometrical characterization of the solutions for some vectorial equations defined by Euclidean norms where they described 'A-Curves' as circles and spheres in the two and three-dimensional Euclidean spaces, respectively. Many semi-module isomorphisms are demonstrated between the direct product of WFC numbers with itself and the direct product of classical Euclidean vector spaces multiplied by itself. Also, in [19], those isomorphisms are classified as 'A-Curves' and related to the WFC ring.

However, WFC numbers and their primary arithmetic operations were programmed by Python [20]. In addition, a special isomorphism transformation function was created in [21] to help move between the WFC set and the real number set. They defined WFC functions, too. After that, Edduweh et al. [22] presented some types of first-order first-degree separable, exact, and linear Weak Fuzzy Complex-Differential Equations (WFC-DEs). In [23], the novel topological space generated by WFC intervals is based on WFC numbers' partially ordered ring structure. Also, it showed the relationship between the WFC intervals and neutrosophic intervals.

Numerous physical phenomena are represented through functions and their derivatives. The first derivative, for instance, signifies the slope or velocity, while the second derivative conveys curvature or acceleration. DEs play a crucial role in various aspects of contemporary life, finding applications in different fields such as biology, medicine, control systems [24], transportation, meteorology [25], telecommunications, and financial modeling [26].

That leads us to pursue an essential area of inquiry that remains underexplored about the DEs in F_J WFC-DEs. Our paper will introduce the second-order WFC-DEs. In Section 2, we will mention some main concepts about WFC numbers, a special isomorphism transformation function, the quadratic WFC equation, WFC functions, and Weak Fuzzy Complex-Ordinary Differential Equations (WFC-ODEs).

In Section 3, we will define the second-order ordinary DEs, introducing the type of linear homogeneous equation with constant coefficients from F_J. Also, we will take the simple case of them with real constant coefficients. We will find the roots of their characteristic equation and construct their general solution with suitable examples. Then, we will give the notion of Weak Fuzzy Complex-Initial Value Problems (WFC-IVPs) of the related second-order DEs in Section 4 and find their particular solutions.

2 | Preliminaries

This section presents important definitions to explain key elements within our analytical framework [10], [13], [21], [22].

Definition 1. The set of WFC numbers was defined as follows:

$$F_J = \{x_0 + x_1 J ;\ x_0, x_1 \in R, J^2 = t \in]0,1[\ \},$$

where 'J' is the WFC operator ($J \notin R$).

Definition 2. Let φ be the transformation function from F_J to $R \times R$, which we define as follows:

$$\varphi: F_J \mapsto R \times R,$$

$$\varphi(x_0 + x_1 J) = \left(x_0 + x_1\left(-\sqrt{t}\right), x_0 + x_1\left(+\sqrt{t}\right)\right) = \left(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}\right),$$

where $J^2 = t \in]0,1[\Rightarrow J = \pm\sqrt{t}$, and $x_0, x_1, y_0, y_1 \in R$ (This map is an isomorphism).

Definition 3. Let $\varphi: F_J \mapsto R \times R$ such that $\varphi(X) = (a, b)$, the inverse function of φ is defined as follows:

$$\varphi^{-1}: R \times R \mapsto F_J,$$

$$\varphi^{-1}(a, b) = \frac{1}{2}[a + b] + \frac{1}{2\sqrt{t}} J[b - a].$$

Definition 4. Let $X = x_0 + x_1 J$, $Y = y_0 + y_1 J \in F_J$, we say that $X \le Y$, if and only if:

$$\begin{cases} x_0 - x_1\sqrt{t} \le y_0 - y_1\sqrt{t}, \\ x_0 + x_1\sqrt{t} \le y_0 + y_1\sqrt{t}. \end{cases}$$

Definition 5. Let $A = a_0 + a_1 J$, $B = b_0 + b_1 J \in F_J$, we define the interval $[A, B]$ if and only if $A \le B$, according to the definition of the partial order relation (≤).

I. If $A \nleq B$, then $[A, B] = \phi$.

II. We can understand [A B] as follows:

$$[A, B] = \left\{C = c_0 + c_1 J \in F_J;\ A \le C \le B\right\}.$$

Theorem 1. Let $aX^2 + bX + c = 0$ be a quadratic WFC equation, where $a = a_0 + a_1 J$, $b = b_0 + b_1 J$ and $c = c_0 + c_1 J \in F_J$, then it is equivalent to:

$$\begin{cases} A_0 {X_0}^2 + B_0 X_0 + C_0 = 0, \\ A_1 {X_1}^2 + B_1 X_1 + C_1 = 0, \end{cases}$$

$a = \varphi^{-1}(A_0, A_1) = \varphi^{-1}\left(a_0 - a_1\sqrt{t}, a_0 + a_1\sqrt{t}\right), A_0 = a_0 - a_1\sqrt{t} \in R, A_1 = a_0 + a_1\sqrt{t} \in R,$

$B = \varphi^{-1}(B_0, B_1) = \varphi^{-1}\left(b_0 - b_1\sqrt{t}, b_0 + b_1\sqrt{t}\right), B_0 = b_0 - b_1\sqrt{t} \in R, B_1 = b_0 + b_1\sqrt{t} \in R,$

$c = \varphi^{-1}(C_0, C_1) = \varphi^{-1}(c_0 - c_1\sqrt{t}, c_0 + c_1\sqrt{t}), C_0 = c_0 - c_1\sqrt{t} \in R, C_1 = c_0 + c_1\sqrt{t} \in R.$

Definition 6. Let $f: F_J \mapsto F_J$ be a WFC function in one variable, where

$\varphi(f(X)) = (f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t})); f_1, f_2: R \mapsto R.$

Then we say:

I. f is continuous on F_J if and only if f_1 and f_2 are continuous on R.

II. f is differentiable on F_J if and only if f_1 and f_2 are differentiable on R, concerning their variables.

III. f is integrable on F_J if only f1 and f_2 are integrable on R.

Definition 7. Let $f: F_J \mapsto F_J$ be a differentiable/integrable function on F_J. We define

I. $f'(X) = \varphi^{-1}(f'_1(x_0 - x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t}))$.

II. $\int f(X).dX = \varphi^{-1}(\int f_1.d(x_0 - x_1\sqrt{t}), \int f_2.d(x_0 + x_1\sqrt{t}))$.

Definition 8. Let Y be a dependent variable, and $Y = f(X)$ represents an unknown WFC function where the independent variable $X = x_0 + x_1 J$ is a WFC variable. We define the WFC DE of n-th order (the highest derivative appearing in the DE) as the following:

$$\mathcal{F}(X, Y, Y', Y'', \dots, Y^{(n)}) = 0, \qquad (1)$$

where $Y^{(n)} = \frac{d^nY}{dX^n}$, $Y = y_0 + y_1 J = \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}) = f(X) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}))$ and $f_1, f_2: R \mapsto R$.

Definition 9. A function $Y = f(X) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}))$ is called the general solution of *Eq. (1)* on $I \subseteq F_J$ if Y is n-times differentiable WFC function on I and satisfies the equation on I, where $f_1, f_2: R \mapsto R$, $I = \varphi^{-1}(I_1 \times I_2) \subseteq F_J$ and $X = \varphi^{-1}(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t})$, $x_0 - x_1\sqrt{t} \in I_1 \subseteq R$, $x_0 + x_1\sqrt{t} \in I_2 \subseteq R$.

We aim to study the second-order DEs that play a pivotal role in modeling systems where the relationship between acceleration and displacement is crucial, such as in mechanics and wave propagation.

3 | Second-Order Ordinary Differential Equations in F_J

We will now propose the definition of the Second-order WFC-DEs. Throughout the discussion, $X = \varphi^{-1}(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}) \in F_J$, $x_0 - x_1\sqrt{t} \in I_1 \subseteq R$, $x_0 + x_1\sqrt{t} \in I_2 \subseteq R$, $I = \varphi^{-1}(I_1 \times I_2) \subseteq F_J$, $Y_0 = y_0 - y_1\sqrt{t} \in R$, $Y_1 = y_0 + y_1\sqrt{t} \in R$, $Y = \varphi^{-1}(Y_0, Y_1) \in F_J$.

Definition 10. The WFC-DE of the second order is written as follows:

$$\mathcal{F}(X, Y, Y', Y'') = 0, \qquad (2)$$

$$\overset{\varphi}{\Leftrightarrow} \begin{cases} \mathcal{F}_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}, y'_0 - y'_1\sqrt{t}, y''_0 - y''_1\sqrt{t}) = 0, & (3-1) \\ \mathcal{F}_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}, y'_0 + y'_1\sqrt{t}, y''_0 + y''_1\sqrt{t}) = 0, & (3-2) \end{cases} \qquad (3)$$

where $\mathcal{F} = \varphi^{-1}(\mathcal{F}_1, \mathcal{F}_2)$, $y_0 - y_1\sqrt{t} = f_1(x_0 - x_1\sqrt{t})$, $y_0 + y_1\sqrt{t} = f_2(x_0 + x_1\sqrt{t})$.

$Y' = \frac{dY}{dX} = f'(X) = \varphi^{-1}(f'_1(x_0 - x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t}))$,

$Y'' = \frac{d^2Y}{dX^2} = f''(X) = \varphi^{-1}(f''_1(x_0 - x_1\sqrt{t}), f''_2(x_0 + x_1\sqrt{t}))$,

and $\frac{d}{dX} = \varphi^{-1}\left(\frac{d}{d(x_0 - x_1\sqrt{t})}, \frac{d}{d(x_0 + x_1\sqrt{t})}\right)$, $\frac{d^2}{dX^2} = \varphi^{-1}\left(\frac{d^2}{d(x_0 - x_1\sqrt{t})^2}, \frac{d^2}{d(x_0 + x_1\sqrt{t})^2}\right)$.

Definition 11. Let $Y_0 = y_0 - y_1\sqrt{t} = f_1(x_0 - x_1\sqrt{t})$ and $Y_1 = y_0 + y_1\sqrt{t} = f_2(x_0 + x_1\sqrt{t})$ are the general solutions to *Eq. (3-1)* on I_1 and *Eq. (3-2)* on I_2, respectively, then

$Y = f(X) = \varphi^{-1}(Y_0, Y_1)$,

is the general solution of *Eq. (2)* on $I \subseteq F_J$.

In other words, $Y = f(X)$ is the general solution of *Eq. (2)* on $I \subseteq F_J$ if $f(X)$ is a twice differentiable WFC function on I.

One of the simplest types of second-order ordinary DEs is the linear homogeneous equation with constant coefficients. We will concentrate on this type, which has many applications in science and engineering.

3.1 | Second-Order WFC Constant-Coefficient Linear Homogeneous Weak Fuzzy Complex Ordinary Differential Equation

Definition 12. The general form of a second-order linear homogeneous WFC ordinary DE with constant coefficients is written as

$$aY'' + bY' + cY = 0, \tag{4}$$

where a, b, c, Y, $a = a_0 + a_1 J$, $b = b_0 + b_1 J$ and $c = c_0 + c_1 J \in F_J$.

Remark 1. Using φ, we find that *Eq. (4)* is equivalent to two second-order linear homogeneous ordinary DEs with constant coefficients in R

$$\text{Eq. (4)} \overset{\varphi}{\Leftrightarrow} \begin{cases} A_0 Y_0'' + B_0 Y_0' + C_0 Y_0 = 0, & (5-1) \\ A_1 Y_1'' + B_1 Y_1' + C_1 Y_1 = 0, & (5-2) \end{cases} \tag{5}$$

where,

$$a = a_0 + a_1 J = \varphi^{-1}(A_0, A_1),\ b = b_0 + b_1 J = \varphi^{-1}(B_0, B_1) \text{ and } c = c_0 + c_1 J = \varphi^{-1}(C_0, C_1) \in F_J,$$

$$A_0 = a_0 - a_1\sqrt{t} \in R,\ A_1 = a_0 + a_1\sqrt{t} \in R,$$

$$B_0 = b_0 - b_1\sqrt{t} \in R,\ B_1 = b_0 + b_1\sqrt{t} \in R, \tag{5}$$

$$C_0 = c_0 - c_1\sqrt{t} \in R,\ C_1 = c_0 + c_1\sqrt{t} \in R.$$

We will use the characteristic equation to solve *Eq. (4)*.

Definition 13. The characteristic equation of the constant-coefficient second-order linear homogeneous WFC-DE *Eq. (4)* is defined as the quadratic WFC equation.

$$am^2 + bm + c = 0, \quad a, b, c \in F_J, \tag{6}$$

which holds the following,

It is equivalent to the corresponding characteristic equations

$$\begin{cases} A_0\lambda^2 + B_0\lambda + C_0 = 0, & (\text{its discriminant } \Delta_1 > 0), \quad (7-1) \\ A_1\mu^2 + B_1\mu + C_1 = 0, & (\text{its discriminant } \Delta_2 > 0). \quad (7-2) \end{cases} \tag{7}$$

It has 4 roots when its discriminant $\Delta = b^2 - 4ac$:

$$l_1 = \varphi^{-1}(\lambda_1, \mu_1), l_2 = \varphi^{-1}(\lambda_2, \mu_2), l_3 = \varphi^{-1}(\lambda_1, \mu_2), l_4 = \varphi^{-1}(\lambda_2, \mu_1).$$

Proof:

Depending on *Remark 1* and *Theorem 1*, we know that the characteristic equations of second-order DEs *Eq. (5-1)*, *Eq. (5-2)* have the forms *Eq. (7-1)*, *Eq. (7-2)*, respectively,

$$\begin{cases} A_0\lambda^2 + B_0\lambda + C_0 = 0, \\ A_1\mu^2 + B_1\mu + C_1 = 0, \end{cases}$$

$$\varphi^{-1}(A_0\lambda^2 + B_0\lambda + C_0, A_1\mu^2 + B_1\mu + C_1) = \varphi^{-1}(0,0)$$

$\Rightarrow \varphi^{-1}(A_0\lambda^2, A_1\mu^2) + \varphi^{-1}(B_0\lambda, B_1\mu) + \varphi^{-1}(C_0, C_1) = \varphi^{-1}(0,0)$
$\Rightarrow \varphi^{-1}(A_0, A_1)\varphi^{-1}(\lambda^2, \mu^2) + \varphi^{-1}(B_0, B_1)\varphi^{-1}(\lambda, \mu) + \varphi^{-1}(C_0, C_1) = \varphi^{-1}(0,0)$
$\Rightarrow\ am^2 + bm + c = 0,$

where

$m = \varphi^{-1}(\lambda, \mu), a, b, c \in F_J,$
$a = \varphi^{-1}(A_0, A_1) = \varphi^{-1}(a_0 - a_1\sqrt{t}, a_0 + a_1\sqrt{t}), A_0 = a_0 - a_1\sqrt{t} \in R, A_1 = a_0 + a_1\sqrt{t} \in R,$
$b = \varphi^{-1}(B_0, B_1) = \varphi^{-1}(b_0 - b_1\sqrt{t}, b_0 + b_1\sqrt{t}), B_0 = b_0 - b_1\sqrt{t} \in R, B_1 = b_0 + b_1\sqrt{t} \in R,$
$c = \varphi^{-1}(C_0, C_1) = \varphi^{-1}(c_0 - c_1\sqrt{t}, c_0 + c_1\sqrt{t}), C_0 = c_0 - c_1\sqrt{t} \in R, C_1 = c_0 + c_1\sqrt{t} \in R.$

Each characteristic equation has its discriminant,

$$\begin{cases} A_0\lambda^2 + B_0\lambda + C_0 = 0, \\ A_1\mu^2 + B_1\mu + C_1 = 0, \end{cases} \xRightarrow[\text{the discriminants}]{} \begin{cases} \Delta_1 = {B_0}^2 - 4A_0C_0, \\ \Delta_2 = {B_1}^2 - 4A_1C_1. \end{cases}$$

We will study the case when $\Delta_1, \Delta_2 > 0 \Longrightarrow$ the corresponding solutions of the characteristic equations *Eq. (7-1)*, *Eq. (7-2)* are real and formed respectively as

$$\begin{cases} \lambda_1 = \dfrac{-B_0 - \sqrt{\Delta_1}}{2A_0}, \lambda_2 = \dfrac{-B_0 + \sqrt{\Delta_1}}{2A_0} \in R, \\ \mu_1 = \dfrac{-B_1 - \sqrt{\Delta_2}}{2A_1}, \mu_2 = \dfrac{-B_1 + \sqrt{\Delta_2}}{2A_1} \in R, \end{cases}$$

$\Rightarrow$ the solutions of the characteristic equations *Eq. (6)* [10].

$$l_1 = \varphi^{-1}(\lambda_1, \mu_1), l_2 = \varphi^{-1}(\lambda_2, \mu_2), l_3 = \varphi^{-1}(\lambda_1, \mu_2), l_4 = \varphi^{-1}(\lambda_2, \mu_1).$$

Theorem 2. Let $aY'' + bY' + cY = 0$ is a second-order constant-coefficients linear homogeneous DE. And its characteristic equation is $am^2 + bm + c = 0$. Then, the general solution of the WFC-ODE has the form

$$Y = M_1 e^{l_1X} + M_2e^{l_2X} + M_3e^{l_3X} + M_4e^{l_4X}.$$

Proof:

We found that

$$aY'' + bY' + c = 0 \overset{\varphi}{\Leftrightarrow} \begin{cases} A_0Y_0'' + B_0Y_0' + C_0Y_0 = 0, \\ A_1Y_1'' + B_1Y_1' + C_1Y_1 = 0, \end{cases} \xRightarrow[\text{the characteristic equations}]{} \begin{cases} A_0\lambda^2 + B_0\lambda + C_0 = 0 \ \ (\text{its discriminant } \Delta_1), \\ A_1\mu^2 + B_1\mu + C_1 = 0 \ \ (\text{its discriminant } \Delta_2). \end{cases}$$

Since $\Delta_1, \Delta_2 > 0$, the characteristic equations have the real different roots, $\begin{cases} \lambda_1, \lambda_2 \in R, \\ \mu_1, \mu_2 \in R. \end{cases}$

Those roots directly influence the behavior of the solutions of the second-order corresponding Ordinary Differential Equations (ODEs) *Eq. (5-1)* and *Eq. (5-2)* [28],

$$\xRightarrow[\text{the general solutions}]{} \begin{cases} Y_0 = He^{\lambda_1X_0} + Ie^{\lambda_2X_0} \\ Y_1 = Ge^{\mu_1X_1} + Ke^{\mu_2X_1} \end{cases},$$ H,I,G, and K, are arbitrary constants.

Using φ^{-1}, we get the general solution of WFC-DE *Eq. (4)*,

$$Y = \varphi^{-1}(Y_0, Y_1) = \varphi^{-1}(He^{\lambda_1 X_0} + Ie^{\lambda_2 X_0}, \qquad Ge^{\mu_1 X_1} + Ke^{\mu_2 X_1})$$

$$= \frac{1}{2}\left(He^{\lambda_1 X_0} + Ie^{\lambda_2 X_0} + Ge^{\mu_1 X_1} + Ke^{\mu_2 X_1}\right) + \frac{1}{2\sqrt{t}}J\left(Ge^{\mu_1 X_1} + Ke^{\mu_2 X_1} - He^{\lambda_1 X_0} - Ie^{\lambda_2 X_0}\right).$$

We can write this structure as $y_1$or y_2,

$$y_1 = \tfrac{1}{2}\left((He^{\lambda_1 X_0} + Ge^{\mu_1 X_1}) + (Ie^{\lambda_2 X_0} + Ke^{\mu_2 X_1})\right) + \tfrac{1}{2\sqrt{t}}J\left((Ge^{\mu_1 X_1} - He^{\lambda_1 X_0}) + (Ke^{\mu_2 X_1} - Ie^{\lambda_2 X_0})\right)$$

$$= \varphi^{-1}\left(He^{\lambda_1 X_0}, Ge^{\mu_1 X_1}\right) + \varphi^{-1}(Ie^{\lambda_2 X_0}, Ke^{\mu_2 X_1})$$

$$= \varphi^{-1}(H,G)\varphi^{-1}\left(e^{\lambda_1 X_0}, e^{\mu_1 X_1}\right) + \varphi^{-1}(I,K)\varphi^{-1}(e^{\lambda_2 X_0}, e^{\mu_2 X_1})$$

$$= \varphi^{-1}(H,G)\, e^{l_1 X} + \varphi^{-1}(I,K)\, e^{l_2 X},$$

or

$$y_2 = \tfrac{1}{2}\left((He^{\lambda_1 X_0} + Ke^{\mu_2 X_1}) + (Ie^{\lambda_2 X_0} + Ge^{\mu_1 X_1})\right) + \tfrac{1}{2\sqrt{t}}J\left((Ge^{\mu_1 X_1} - Ie^{\lambda_2 X_0}) + (Ke^{\mu_2 X_1} - He^{\lambda_1 X_0})\right)$$

$$= \varphi^{-1}\left(He^{\lambda_1 X_0}, Ke^{\mu_2 X_1}\right) + \varphi^{-1}(Ie^{\lambda_2 X_0}, Ge^{\mu_1 X_1})$$

$$= \varphi^{-1}(H,K)\varphi^{-1}\left(e^{\lambda_1 X_0}, e^{\mu_2 X_1}\right) + \varphi^{-1}(I,G)\varphi^{-1}(e^{\lambda_2 X_0}, e^{\mu_1 X_1})$$

$$= \varphi^{-1}(H,K)\, e^{l_3 X} + \varphi^{-1}(I,G)\, e^{l_4 X}.$$

The linearity of equation *Eq. (4)* means that if we have two solutions y_1 and y_2, then any linear combination of y_1 and y_2 is also a solution of this equation, i.e.

$$Y(X) = p_1 y_1(X) + p_2 y_2(X)$$

$$= p_1(\varphi^{-1}(H,G)\, e^{l_1 X} + \varphi^{-1}(I,K)\, e^{l_2 X}) + p_2(\varphi^{-1}(H,K)\, e^{l_3 X} + \varphi^{-1}(I,G)\, e^{l_4 X})$$

$$= M_1\, e^{l_1 X} + M_2 e^{l_2 X} + M_3 e^{l_3 X} + M_4 e^{l_4 X},$$

is a solution for any choice of constants $p_1, p_2, M_1 = p_1\varphi^{-1}(H,G), M_2 = p_1\varphi^{-1}(I,K), M_3 = p_2\varphi^{-1}(H,K), M_4 = p_2\varphi^{-1}(I,G)$.

Now, let's verify that the solution $Y(X)$ satisfies *Eq. (4)*.

$$a\left(M_1\, e^{l_1 X} + M_2 e^{l_2 X} + M_3 e^{l_3 X} + M_4 e^{l_4 X}\right)'' + b\left(M_1\, e^{l_1 X} + M_2 e^{l_2 X} + M_3 e^{l_3 X} + M_4 e^{l_4 X}\right)' + c\left(M_1\, e^{l_1 X} + M_2 e^{l_2 X} + M_3 e^{l_3 X} + M_4 e^{l_4 X}\right)$$

$$= \left[a\left(M_1\, e^{l_1 X} + M_2 e^{l_2 X}\right)'' + b\left(M_1\, e^{l_1 X} + M_2 e^{l_2 X}\right)' + c\left(M_1\, e^{l_1 X} + M_2 e^{l_2 X}\right)\right]$$

$$+ \left[a\left(M_3 e^{l_3 X} + M_4 e^{l_4 X}\right)'' + b\left(M_3 e^{l_3 X} + M_4 e^{l_4 X}\right)' + c\left(M_3 e^{l_3 X} + M_4 e^{l_4 X}\right)\right]$$

$$= p_1[a\left(\varphi^{-1}(H,G)\, e^{l_1 X} + \varphi^{-1}(I,K)\, e^{l_2 X}\right)'' + b\left(\varphi^{-1}(H,G)\, e^{l_1 X} + \varphi^{-1}(I,K)\, e^{l_2 X}\right)' + c\left(\varphi^{-1}(H,G)\, e^{l_1 X} + \varphi^{-1}(I,K)\, e^{l_2 X}\right)]$$

$$+ p_2[a\left(\varphi^{-1}(H,K)\, e^{l_3 X} + \varphi^{-1}(I,G)\, e^{l_4 X}\right)'' + b\left(\varphi^{-1}(H,K)\, e^{l_3 X} + \varphi^{-1}(I,G)\, e^{l_4 X}\right)' + c\left(\varphi^{-1}(H,K)\, e^{l_3 X} + \varphi^{-1}(I,G)\, e^{l_4 X}\right)]$$

$$= p_1[a{y_1}'' + b{y_1}' + cy_1] + p_2[a{y_2}'' + b{y_2}' + cy_2]$$

$= p_1(0) + p_2(0)$ (because y_1 and y_2 are solutions of *Eq.* (4))

$= 0,$

where $m = \varphi^{-1}(\lambda, \mu), l_1 = \varphi^{-1}(\lambda_1, \mu_1), l_2 = \varphi^{-1}(\lambda_2, \mu_2), l_3 = \varphi^{-1}(\lambda_1, \mu_2), l_4 = \varphi^{-1}(\lambda_2, \mu_1) \in F_J$.

$a = a_0 + a_1 J = \varphi^{-1}(A_0, A_1) = \varphi^{-1}(a_0 - a_1\sqrt{t}, a_0 + a_1\sqrt{t}), A_0 = a_0 - a_1\sqrt{t} \in R, A_1 = a_0 + a_1\sqrt{t} \in R,$

$b = b_0 + b_1 J = \varphi^{-1}(B_0, B_1) = \varphi^{-1}(b_0 - b_1\sqrt{t}, b_0 + b_1\sqrt{t}), B_0 = b_0 - b_1\sqrt{t} \in R, B_1 = b_0 + b_1\sqrt{t} \in R,$

$c = c_0 + c_1 J = \varphi^{-1}(C_0, C_1) = \varphi^{-1}(c_0 - c_1\sqrt{t}, c_0 + c_1\sqrt{t}), C_0 = c_0 - c_1\sqrt{t} \in R, C_1 = c_0 + c_1\sqrt{t} \in R.$

Example 1. $(3 + 2J)\, Y'' + J\, Y' - (15 + \frac{45}{2}J)Y = 0.$

Its characteristic equation is

$$(3 + 2J)m^2 + Jm - (15 + \frac{45}{2}J) = 0.$$

The two related DEs of the WFC-DE are [10].

$$\begin{cases}(3 - 2\sqrt{t})Y_0'' - \sqrt{t}\,Y_0' - \left(15 - \frac{45}{2}\sqrt{t}\right)Y_0 = 0\\ (3 + 2\sqrt{t})Y_1'' + \sqrt{t}\,Y_1' - \left(15 + \frac{45}{2}\sqrt{t}\right)Y_1 = 0\end{cases} \xRightarrow[J^2=t=\frac{1}{4}]{\text{for}} \begin{cases}2Y_0'' - \frac{1}{2}Y_0' - \frac{15}{4}Y_0 = 0,\\ 4Y_1'' + \frac{1}{2}Y_1' - \frac{105}{4}Y_1 = 0,\end{cases}$$

$$\xRightarrow[\text{the characteristic equations}]{} \begin{cases}2\lambda^2 - \frac{1}{2}\lambda - \frac{15}{4} = 0 \;\left(\text{its discriminant } \Delta_1 = \frac{121}{4} > 0\right)\\ 4\mu^2 + \frac{1}{2}\mu - \frac{105}{4} = 0 \;\left(\text{its discriminant } \Delta_2 = \frac{1681}{4} > 0\right)\end{cases} \Rightarrow$$

$$\begin{cases}\lambda_1 = -\frac{5}{4}, \lambda_2 = \frac{3}{2} \in R,\\ \mu_1 = -\frac{11}{4}, \mu_2 = \frac{5}{2} \in R,\end{cases}$$

$$\xRightarrow[\text{the general solutions}]{} \begin{cases}Y_0 = He^{-\frac{5}{4}X_0} + Ie^{\frac{3}{2}X_0},\\ Y_1 = Ge^{-\frac{11}{4}X_1} + Ke^{\frac{5}{2}X_1},\end{cases}$$

H, I, G, K are arbitrary constants.

Thus

$$l_1 = \varphi^{-1}(\lambda_1, \mu_1) = \varphi^{-1}\left(-\frac{5}{4}, -\frac{11}{4}\right) = \frac{1}{2}\left(-\frac{5}{4} - \frac{11}{4}\right) + J\left(-\frac{11}{4} + \frac{5}{4}\right) = -2 - \frac{3}{2}J,$$

$$l_2 = \varphi^{-1}(\lambda_2, \mu_2) = \varphi^{-1}\left(\frac{3}{2}, \frac{5}{2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{5}{2}\right) + J\left(\frac{5}{2} - \frac{3}{2}\right) = 2 + J,$$

$$l_3 = \varphi^{-1}(\lambda_1, \mu_2) = \varphi^{-1}\left(-\frac{5}{4}, \frac{5}{2}\right) = \frac{1}{2}\left(-\frac{5}{4} + \frac{5}{2}\right) + J\left(\frac{5}{2} + \frac{5}{4}\right) = \frac{5}{8} + \frac{15}{4}J,$$

$$l_4 = \varphi^{-1}(\lambda_2, \mu_1) = \varphi^{-1}\left(\frac{3}{2}, -\frac{11}{4}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{11}{4}\right) + J\left(-\frac{11}{4} - \frac{3}{2}\right) = -\frac{5}{8} - \frac{17}{4}J.$$

Then, the general solution of WFC-DE

$$y = m_1\, e^{(-2-\frac{3}{2}J)x} + m_2 e^{(2+J)x} + m_3 e^{(\frac{5}{8}+\frac{15}{4}J)x} + m_4 e^{\left(-\frac{5}{8}-\frac{17}{4}J\right)x},$$

where $p_1, p_2, M_1 = p_1\varphi^{-1}(H, G), M_2 = p_1\varphi^{-1}(I, K), M_3 = p_2\varphi^{-1}(H, K), M_4 = p_2\varphi^{-1}(I, G)$ are arbitrary constants.

Now, we will study the simple form of this equation with a real constant.

3.2 | Second-Order Linear Homogeneous WFC-ODE with Real Constant Coefficients

The real constant-coefficient second-order linear homogeneous WFC-DE is adopted as

$$aY'' + bY' + cY = 0, \quad a, b, c \in R.$$

Its characteristic equation $am^2 + bm + c = 0$, where its discriminant $\Delta = b^2 - 4ac$ and its roots are $l_1, l_2, l_3, l_4 \in F_J$.

Since the coefficients of the quadratic WFC equation *Eq. (6)* are real a, b and $c \in R$, the second-order DEs *Eq. (5-1)*, *Eq. (5-2)* have the following characteristic equations *Eq. (8-1)*, and *Eq. (8-2)*:

$$\begin{cases} aY_0'' + bY_0' + cY_0 = 0, \\ aY_1'' + bY_1' + cY_1 = 0, \end{cases} \xrightarrow[\text{equations}]{\text{the characteristic}} \begin{cases} a\lambda^2 + b\lambda + c = 0, & (8-1) \\ a\mu^2 + b\mu + c = 0. & (8-2) \end{cases} \quad (8)$$

Eq. (8-1), and *Eq. (8-2)* have the discriminants $\Delta_1 = \Delta_2 = \Delta = b^2 - 4ac > 0$.

Then, we will get the real roots of *Eq. (8-1)*, and *Eq. (8-2)*:

$$\lambda_1 = \mu_1 = \frac{-b-\sqrt{\Delta}}{2a} \lambda_2 = \mu_2 = \frac{-b+\sqrt{\Delta}}{2a},$$

and

$$\lambda_2 = \mu_2 = \frac{-b+\sqrt{\Delta}}{2a}.$$

Example 2.

$$Y'' - 4Y' + 3Y = 0. \quad (9)$$

The characteristic equation of WFC-DE *Eq. (9)* is

$$m^2 - 4m + 3 = 0. \quad (10)$$

The two related DEs of *Eq. (9)* are

$$\begin{cases} Y_0'' - 4Y_0' + 3Y_0 = 0, \\ Y_1'' - 4Y_1' + 3Y_1 = 0, \end{cases} \xrightarrow[\text{equations}]{\text{the characteristic}} \begin{cases} \lambda^2 - 4\lambda + 3 = 0 \\ \mu^2 - 4\mu + 3 = 0 \end{cases} \Rightarrow \Delta_1 = \Delta_2 > 0 \Rightarrow$$

$$\begin{cases} \lambda_1 = 3, \lambda_2 = 1 \in R, \\ \mu_1 = 3, \mu_2 = 1 \in R, \end{cases}$$

$$\xrightarrow[\text{solutions}]{\text{The general}} \begin{cases} Y_0 = He^{3X_0} + Ie^{X_0}, \\ Y_1 = Ge^{3X_1} + Ke^{X_1}, \end{cases}$$

H, I, G, K are arbitrary constants.

Then the roots of *Eq. (10)* are

$$l_1 = \varphi^{-1}(\lambda_1, \mu_1) = \varphi^{-1}(3,3) = \frac{1}{2}(3+3) + \frac{1}{2\sqrt{t}}J(3-3) = 3,$$

$$l_2 = \varphi^{-1}(\lambda_2, \mu_2) = \varphi^{-1}(1,1) = \frac{1}{2}(1+1) + \frac{1}{2\sqrt{t}}J(1-1) = 1,$$

$$l_3 = \varphi^{-1}(\lambda_1, \mu_2) = \varphi^{-1}(3,1) = \frac{1}{2}(3+1) + \frac{1}{2\sqrt{t}}J(1-3) = 2 - \frac{1}{\sqrt{t}}J,$$

$$l_4 = \varphi^{-1}(\lambda_2, \mu_1) = \varphi^{-1}(3,1) = \frac{1}{2}(1+3) + \frac{1}{2\sqrt{t}}J(3-1) = 2 + \frac{1}{\sqrt{t}}J.$$

Then, the general solution of *Eq. (9)*

$$Y = M_1\, e^{3X} + M_2 e^{X} + M_3 e^{(2-\frac{1}{\sqrt{t}}J)X} + M_4 e^{(2+\frac{1}{\sqrt{t}}J)X},$$

where $p_1, p_2, M_1 = p_1\varphi^{-1}(H, G), M_2 = p_1\varphi^{-1}(I, K), M_3 = p_2\varphi^{-1}(H, K), M_4 = p_2\varphi^{-1}(I, G)$ are arbitrary constants.

We know that arbitrary constants in any general solution of a DE are determined by specifying initial conditions. For a second-order DE in F_J, we should have two WFC initial conditions to determine the two arbitrary constants p_1 and p_2, which are equivalent to four initial conditions in R.

4 | Second-Order Constant-Coefficient Linear Homogeneous Weak Fuzzy Complex Initial Value Problems

Definition 14. The second-order WFC-Initial Value Problem consists of finding the solution of a WFC-ODE given two initial conditions.

$$\begin{cases} \mathcal{F}(X, Y, Y', Y'') = 0, \\ Y(\alpha) = \nu, \qquad Y'(\alpha) = \omega, \quad \text{(the initial conditions)} \end{cases}, \alpha \le X \le \beta, \tag{11}$$

where a fixed WFC value $(\alpha, \nu), (\alpha, \omega) \in F_J \times F_J$ are given.

Using φ function for *Eq. (11)*, we get

$$\begin{cases} \varphi^{-1}\big(\mathcal{F}_1(X_0, Y_0, Y_0', Y_0''), \mathcal{F}_2(X_1, Y_1, Y_1', Y_1'')\big) = 0 \\ \qquad \varphi^{-1}\big(Y_0(\alpha_0), Y_1(\alpha_1)\big) = \varphi^{-1}(\nu_0, \nu_1), \varphi^{-1}\big(Y_0'(\alpha_0), Y_1'(\alpha_1)\big) = \varphi^{-1}(\omega_0, \omega_1) \end{cases}$$

$$\Rightarrow \begin{cases} \begin{cases} \mathcal{F}_1(X_0, Y_0, Y_0', Y_0'') = 0, \\ Y_0(\alpha_0) = \nu_0, Y_0'^{(\alpha_0)} = \omega_0, \end{cases} & \alpha_0 \le X_0 \le \beta_0, \quad (12) \Rightarrow \text{ its solution is } Y_0(X_0), \\ \begin{cases} \mathcal{F}_2(X_1, Y_1, Y_1', Y_1'') = 0, \\ Y_1(\alpha_1) = \nu_1, Y_1'^{(\alpha_1)} = \omega_1, \end{cases} & \alpha_1 \le X_1 \le \beta_1, \quad (13) \Rightarrow \text{ its solution is } Y_1(X_1). \end{cases}$$

Therefore, the particular solution of the second-order WFC-IVP *Eq. (11)* is a structure of the related solutions of the second-order initial-value problems *Eq. (12)* and *Eq. (13)*,

$$Y = \varphi^{-1}\big(Y_0(X_0), Y_1(X_1)\big),$$

where

$X = \varphi^{-1}(X_0, X_1) \in [\alpha, \beta] \subseteq F_J, X_0 = x_0 - x_1\sqrt{t} \in [\alpha_0, \beta_0] \subseteq R, X_1 = x_0 + x_1\sqrt{t} \in [\alpha_1, \beta_1] \subseteq R,$

$Y = \varphi^{-1}(Y_0, Y_1),\ Y_0 = y_0 - y_1\sqrt{t},\ Y_1 = y_0 + y_1\sqrt{t},$

$\alpha = \varphi^{-1}(\alpha_0, \alpha_1), \beta = \varphi^{-1}(\beta_0, \beta_1), \nu = \varphi^{-1}(\nu_0, \nu_1),\ \omega = \varphi^{-1}(\omega_0, \omega_1).$

We get the particular solution when we substitute the initial conditions in the general solution.

Definition 15. The constant-coefficient second-order linear homogeneous WFC initial value problem (WFC-IVP) is formed as

$$\begin{cases} aY'' + bY' + cY = 0, \\ Y(\alpha) = \nu, Y'(\alpha) = \omega, \end{cases} \quad \alpha \le X \le \beta, \tag{12}$$

$a = \varphi^{-1}(A_0, A_1), \qquad A_0 = a_0 - a_1\sqrt{t},\ A_1 = a_0 + a_1\sqrt{t},$
$b = \varphi^{-1}(B_0, B_1), \qquad B_0 = b_0 - b_1\sqrt{t},\ B_1 = b_0 + b_1\sqrt{t},$
$c = \varphi^{-1}(C_0, C_1), \qquad C_0 = c_0 - c_1\sqrt{t},\ C_1 = c_0 + c_1\sqrt{t}.$

Remark 2. Using φ, we find that *Eq. (12)* is equivalent to two initial value problems of the same type in R,

$$\text{Eq. (12)} \overset{\varphi}{\Rightarrow} \begin{cases} \begin{cases} A_0 Y_0'' + B_0 Y_0' + C_0 Y_0 = 0, \\ Y_0(\alpha_0) = \nu_0, Y_0'(\alpha_0) = \omega_0, \end{cases} & \alpha_0 \le X_0 \le \beta_0, \\ \begin{cases} A_1 Y_1'' + B_1 Y_1' + C_1 Y_1 = 0, \\ Y_1(\alpha_1) = \nu_1, Y_1'^{(\alpha_1)} = \omega_1, \end{cases} & \alpha_1 \le X_1 \le \beta_1, \end{cases} \tag{13}$$

where $p_1, p_2, M_1 = p_1\varphi^{-1}(H, G), M_2 = p_1\varphi^{-1}(I, K), M_3 = p_2\varphi^{-1}(H, K), M_4 = p_2\varphi^{-1}(I, G)$ are determined constants.

Remark 3. Steps to get the particular solution of the second-order WFC-IVP *Eq. (12)*:

1. Solving the second-order initial-value problems IVPs *Eq. (13)*, we get the specific solutions.

$$\begin{cases} Y_0 = \tilde{H}e^{3X_0} + \tilde{I}e^{X_0}, \\ Y_1 = \tilde{G}e^{3X_1} + \tilde{K}e^{X_1,} \end{cases}$$

$\tilde{H}, \tilde{I}, \tilde{G}, \tilde{K}$ are determined constants.

2. Making a structure of the related particular solutions Y_0 and Y_1 of the first step.
3. $Y(X) = p_1(\varphi^{-1}(\tilde{H}, \tilde{G})\, e^{l_1X} + \varphi^{-1}(\tilde{I}, \tilde{K})\, e^{l_2X}) + p_2(\varphi^{-1}(\tilde{H}, \tilde{K})\, e^{l_3X} + \varphi^{-1}(\tilde{I}, \tilde{G})\, e^{l_4X})$.
4. Substituting the initial conditions $Y(\alpha) = \nu, Y'(\alpha) = \omega$ to have two WFC linear equations.
5. Solving the two WFC linear equations to get values of p_1, p_2.
6. Substituting p_1 and p_2 in the structure of the second step.

In other words, to find the arbitrary constants p_1 〖,p〗_2, 〖〖M_1=p_1 ϕ〗^(-1) (H,G),M_2=p_1 ϕ^(-1) (I,K),M〗_3=p_2 ϕ^(-1) (H,K),M_4=p_2 ϕ^(-1) (I,G), we have to solve the six following equations:

Y_0 (α_0)=ν_0,Y_0^'(α_0) =ω_0 (which give the values of H and I).

Y_1 (α_1)=ν_1,Y_1'(α_1)=ω_1 (which give the values of G and K).

Then

Y(α)=ν ,Y^' (α)=ω (which give the values of p_1and p_2).

Example 3. Second-order linear Homogeneous WFC-IVP with WFC Constant Coefficients.

$$\begin{cases} (3 + 2J)\, Y'' + J\, Y' - (15 + \frac{45}{2}J)Y = 0, & 0 \le X \le 1 + J. \\ Y(0) = 0, \;\; Y'(0) = 1 + J, \end{cases} \tag{14}$$

This Initial Value Problem (IVP) is equivalent to the two following IVPs:

$$\text{Eq. (14)} \overset{\varphi}{\Rightarrow} \begin{cases} \begin{cases} (3 - 2\sqrt{t})Y_0'' - \sqrt{t}\, Y_0' - \left(15 - \frac{45}{2}\sqrt{t}\right) Y_0 = 0, \\ Y_0(0) = 0, Y_0'^{(0)} = 1 - \sqrt{t}, \end{cases} & 0 \le X_0 \le 1 - \sqrt{t} \\ \begin{cases} (3 + 2\sqrt{t})Y_1'' + \sqrt{t}\, Y_1' - \left(15 + \frac{45}{2}\sqrt{t}\right) Y_1 = 0, \\ Y_1(0) = 0, Y_1'(0) = 1 + \sqrt{t}, \end{cases} & 0 \le X_1 \le 1 + \sqrt{t} \end{cases}$$

$$\xrightarrow[J^2 = t = \frac{1}{4}]{\text{for}} \begin{cases} \begin{cases} 2Y_0'' - \frac{1}{2}\, Y_0' - \frac{15}{4} Y_0 = 0, \\ Y_0(0) = 0, Y_0'^{(0)} = \frac{1}{2}, \end{cases} & 0 \le X_0 \le \frac{1}{2} \end{cases} \tag{15}$$

$$\xrightarrow[\mathrm{J^2=t=\frac{1}{4}}]{\text{for}} \left\{ \begin{cases} 4Y_1'' + \frac{1}{2}Y_1' - \frac{105}{4}Y_1 = 0, \\ Y_1(0) = 0, Y_1'^{(0)} = \frac{3}{2}, \end{cases} \right. \quad 0 \le X_1 \le \frac{3}{2}, \tag{16}$$

$$\xrightarrow[\substack{\text{solutions}\\ \text{(Example1)}}]{\text{the general}} \begin{cases} Y_0 = He^{-\frac{5}{4}X_0} + Ie^{\frac{3}{2}X_0}, & Y_0' = -\frac{5}{4}He^{-\frac{5}{4}X_0} + \frac{3}{2}Ie^{\frac{3}{2}X_0}, \\ Y_1 = Ge^{-\frac{11}{4}X_1} + Ke^{\frac{5}{2}X_1}, & Y_1' = -\frac{11}{4}Ge^{-\frac{11}{4}X_1} + \frac{5}{2}Ke^{\frac{5}{2}X_1}. \end{cases}$$

After substituting the corresponding initial conditions to find the values of H, I, G, K:

$$\left\{ \begin{array}{l} \begin{cases} 0 = H + I, \\ \frac{1}{2} = -\frac{5}{4}H + \frac{3}{2}I, \end{cases} \xrightarrow{\text{solving this system}} \begin{cases} \tilde{H} = \frac{-2}{11} \\ \tilde{I} = \frac{2}{11} \end{cases} \xrightarrow[\substack{\text{solution}\\ \text{for IVP Eq. (15)}}]{\text{the particular}} Y_0 = \frac{-2}{11}e^{-\frac{5}{4}X_0} + \frac{2}{11}e^{\frac{3}{2}X_0}, \\ \begin{cases} 0 = G + K, \\ \frac{3}{2} = -\frac{11}{4}G + \frac{5}{2}K, \end{cases} \xrightarrow{\text{solving this system}} \begin{cases} \tilde{G} = \frac{-6}{21} \\ \tilde{K} = \frac{6}{21} \end{cases} \xrightarrow[\substack{\text{solution}\\ \text{for IVP Eq. (16)}}]{\text{the particular}} Y_1 = \frac{-6}{21}e^{-\frac{11}{4}X_1} + \frac{6}{21}e^{\frac{5}{2}X_1}, \end{array} \right.$$

$$l_1 = \varphi^{-1}(\lambda_1, \mu_1) = \varphi^{-1}\left(-\frac{5}{4}, -\frac{11}{4}\right) = \frac{1}{2}\left(-\frac{5}{4} - \frac{11}{4}\right) + J\left(-\frac{11}{4} + \frac{5}{4}\right) = -2 - \frac{3}{2}J,$$

$$l_2 = \varphi^{-1}(\lambda_2, \mu_2) = \varphi^{-1}\left(\frac{3}{2}, \frac{5}{2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{5}{2}\right) + J\left(\frac{5}{2} - \frac{3}{2}\right) = 2 + J,$$

$$l_3 = \varphi^{-1}(\lambda_1, \mu_2) = \varphi^{-1}\left(-\frac{5}{4}, \frac{5}{2}\right) = \frac{1}{2}\left(-\frac{5}{4} + \frac{5}{2}\right) + J\left(\frac{5}{2} + \frac{5}{4}\right) = \frac{5}{8} + \frac{15}{4}J,$$

$$l_4 = \varphi^{-1}(\lambda_2, \mu_1) = \varphi^{-1}\left(\frac{3}{2}, -\frac{11}{4}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{11}{4}\right) + J\left(-\frac{11}{4} - \frac{3}{2}\right) = -\frac{5}{8} - \frac{17}{4}J,$$

$$\varphi^{-1}\left(\frac{-2}{11}, \frac{-6}{21}\right) = \frac{1}{2}\left(\frac{-2}{11} - \frac{6}{21}\right) + J\left(\frac{-6}{21} - \frac{-2}{11}\right) = -\frac{18}{77} - \frac{8}{77}J,$$

$$\varphi^{-1}\left(\frac{2}{11}, \frac{6}{21}\right) = \frac{1}{2}\left(\frac{2}{11} + \frac{6}{21}\right) + J\left(\frac{6}{21} - \frac{2}{11}\right) = \frac{18}{77} + \frac{8}{77}J,$$

$$\varphi^{-1}\left(\frac{-2}{11}, \frac{6}{21}\right) = \frac{1}{2}\left(\frac{-2}{11} + \frac{6}{21}\right) + J\left(\frac{6}{21} + \frac{2}{11}\right) = \frac{4}{77} + \frac{36}{77}J,$$

$$\varphi^{-1}\left(\frac{2}{11}, \frac{-6}{21}\right) = \frac{1}{2}\left(\frac{2}{11} + \frac{-6}{21}\right) + J\left(\frac{-6}{21} - \frac{2}{11}\right) = -\frac{4}{77} - \frac{36}{77}J.$$

Then, the general solution of *Eq. (14)* (from *Example 1*)

$$\begin{aligned} Y(X) &= M_1\, e^{(-2-\frac{3}{2}J)X} + M_2 e^{(2+J)X} + M_3 e^{(\frac{5}{8}+\frac{15}{4}J)X} + M_4 e^{(-\frac{5}{8}-\frac{17}{4}J)X} \\ &= p_1(\varphi^{-1}(\tilde{H}, \tilde{G})\, e^{l_1X} + \varphi^{-1}(\tilde{I}, \tilde{K})\, e^{l_2X}) + p_2(\varphi^{-1}(\tilde{H}, \tilde{K})\, e^{l_3X} + \varphi^{-1}(\tilde{I}, \tilde{G})\, e^{l_4X}) \\ &= p_1(\varphi^{-1}\left(\frac{-2}{11}, \frac{-6}{21}\right) e^{l_1X} + \varphi^{-1}\left(\frac{2}{11}, \frac{6}{21}\right) e^{l_2X}) + p_2(\varphi^{-1}\left(\frac{-2}{11}, \frac{6}{21}\right) e^{l_3X} + \\ &\quad \varphi^{-1}\left(\frac{2}{11}, \frac{-6}{21}\right) e^{l_4X}) \\ &= p_1\left(\left(-\frac{18}{77} - \frac{8}{77}J\right) e^{\left(-2-\frac{3}{2}J\right)X} + \left(\frac{18}{77} + \frac{8}{77}J\right) e^{(2+J)X}\right) + p_2\left(\left(\frac{4}{77} + \frac{36}{77}J\right) e^{\left(\frac{5}{8}+\frac{15}{4}J\right)X} + \right. \\ &\quad \left.\left(-\frac{4}{77} - \frac{36}{77}J\right) e^{\left(-\frac{5}{8}-\frac{17}{4}J\right)X}\right), \end{aligned}$$

$$Y'(X) = p_1\left(\left(-2-\frac{3}{2}J\right)\left(-\frac{18}{77}-\frac{8}{77}J\right)\, e^{\left(-2-\frac{3}{2}J\right)X} + (2+J)\left(\frac{18}{77}+\frac{8}{77}J\right)\, e^{(2+J)\,X}\right) +$$
$$p_2\left(\left(\frac{5}{8}+\frac{15}{4}J\right)\left(\frac{4}{77}+\frac{36}{77}J\right)\, e^{\left(\frac{5}{8}+\frac{15}{4}J\right)X} + \left(-\frac{5}{8}-\frac{17}{4}J\right)\left(-\frac{4}{77}-\frac{36}{77}J\right)\, e^{\left(-\frac{5}{8}-\frac{17}{4}J\right)X}\right).$$

Substituting the initial conditions: $Y(0) = 0$ in the general solution Y, and $Y'(0) = 1 + J$ in the first derivative of the general solution Y', we get

$$\begin{cases} 0 = p_1\left(\left(-\frac{18}{77}-\frac{8}{77}J\right) + \left(\frac{18}{77}+\frac{8}{77}J\right)\right) + p_2\left(\left(\frac{4}{77}+\frac{36}{77}J\right) + \left(-\frac{4}{77}-\frac{36}{77}J\right)\right), \\ 1+J = p_1\left(\left(2+\frac{3}{2}J\right)\left(\frac{18}{77}+\frac{8}{77}J\right) + (2+J)\left(\frac{18}{77}+\frac{8}{77}J\right)\right) + p_2\left(\left(\frac{5}{8}+\frac{15}{4}J\right)\left(\frac{4}{77}+\frac{36}{77}J\right) + \left(\frac{5}{8}+\frac{17}{4}J\right)\left(\frac{4}{77}+\frac{36}{77}J\right)\right) \end{cases}$$

$$\Rightarrow \begin{cases} 0 = 0p_1 + 0p_2 \\ 1+J = \left(\frac{18}{77}+\frac{8}{77}J\right)\left(4+\frac{5}{2}J\right)p_1 + \left(\frac{4}{77}+\frac{36}{77}J\right)\left(\frac{10}{8}+\frac{32}{4}J\right)p_2 \end{cases}.$$

We find that those two equations are satisfied when

$$p_1 = \frac{\left[1+J-\left(\frac{4}{77}+\frac{36}{77}J\right)\left(\frac{10}{8}+\frac{32}{4}J\right)p_2\right]}{\left[\left(\frac{18}{77}+\frac{8}{77}J\right)\left(4+\frac{5}{2}J\right)\right]} = P(p_2).$$

Then the solution of *Eq. (15)* is in terms of p_2 for any p_1:

$$Y(X) = P(p_2)\left(\left(-\frac{18}{77}-\frac{8}{77}J\right)\, e^{\left(-2-\frac{3}{2}J\right)X} + \left(\frac{18}{77}+\frac{8}{77}J\right)\, e^{(2+J)\,X}\right) + p_2\left(\left(\frac{4}{77}+\right.\right.$$
$$\left.\left.\frac{36}{77}J\right)\, e^{\left(\frac{5}{8}+\frac{15}{4}J\right)X} + \left(-\frac{4}{77}-\frac{36}{77}J\right)\, e^{\left(-\frac{5}{8}-\frac{17}{4}J\right)X}\right).$$

Now, we will solve a simple form, 'the real constant-coefficient second-order linear homogeneous WFC ordinary DE.'

Example 4. Second-order linear Homogeneous WFC-IVP with Real Constant Coefficients.

$$\begin{cases} Y'' - 4Y' + 3Y = 0, \\ Y(0) = 3,\ \ Y'(0) = 5, \end{cases} \quad 0 \le X \le 10. \tag{17}$$

This IVP is equivalent to the two following IVPs:

$$\text{Eq. (17)} \overset{\varphi}{\Rightarrow} \begin{cases} \begin{cases} Y_0'' - 4{Y_0}' + 3Y_0 = 0, \\ Y_0(0) = 3, {Y_0}'(0) = 5, \end{cases} & 0 \le X_0 \le 10, \\ \begin{cases} Y_1'' - 4Y_1' + 3Y_1 = 0, \\ Y_1(0) = 3, {Y_1}'(0) = 5, \end{cases} & 0 \le X_1 \le 10, \end{cases}$$

$$\xrightarrow[\text{Example 2}]{\text{the general solutions from}} \begin{cases} Y_0 = He^{3X_0} + Ie^{X_0}, {Y_0}' = 3He^{3X_0} + Ie^{X_0}, \\ Y_1 = Ge^{3X_1} + Ke^{X_1}, Y_1' = 3Ge^{3X_1} + Ke^{X_1}. \end{cases}$$

After substituting the corresponding initial conditions,

$$\begin{cases} \begin{cases} 3 = H + I \\ 5 = 3H + I \end{cases} \Longrightarrow \begin{cases} \tilde{H} = 1 \\ \tilde{I} = 2 \end{cases} \Rightarrow Y_0 = e^{3X_0} + 2e^{X_0}, \\ \begin{cases} 3 = G + K \\ 5 = 3G + K \end{cases} \Longrightarrow \begin{cases} \tilde{G} = 1 \\ \tilde{K} = 2 \end{cases} \Rightarrow Y_1 = e^{3X_1} + 2e^{X_1}. \end{cases}$$

The general solution of *Eq. (17)*,

$$Y(X) = p_1(\varphi^{-1}(\tilde{H},\tilde{G})\, e^{l_1X} + \varphi^{-1}(\tilde{I},\tilde{K})\, e^{l_2X}) + p_2(\varphi^{-1}(\tilde{H},\tilde{K})\, e^{l_3X} + \varphi^{-1}(\tilde{I},\tilde{G})\, e^{l_4X})$$
$$= p_1(\varphi^{-1}(1,1)\, e^{l_1X} + \varphi^{-1}(2,2)\, e^{l_2X}) + p_2(\varphi^{-1}(1,2)\, e^{l_3X} + \varphi^{-1}(2,1)\, e^{l_4X})$$

$$\varphi^{-1}(1,1) = \frac{1}{2}(1+1) + \frac{1}{2\sqrt{t}}J(1-1) = 1,$$

$$\varphi^{-1}(2,2) = 2,$$

$$\varphi^{-1}(1,2) = \frac{1}{2}(1+2) + \frac{1}{2\sqrt{t}}J(2-1) = \frac{3}{2} + \frac{1}{2\sqrt{t}}J,$$
$$\varphi^{-1}(2,1) = \frac{1}{2}(2+1) + \frac{1}{2\sqrt{t}}J(1-2) = \frac{3}{2} - \frac{1}{2\sqrt{t}}J,$$
$$l_1 = \varphi^{-1}(\lambda_1,\mu_1) = 3,$$
$$l_2 = \varphi^{-1}(\lambda_2,\mu_2) = 1,$$
$$l_3 = \varphi^{-1}(\lambda_1,\mu_2) = 2 - \frac{1}{\sqrt{t}}J,$$
$$l_4 = \varphi^{-1}(\lambda_2,\mu_1) = 2 + \frac{1}{\sqrt{t}}J.$$

Then

$$Y(X) = p_1\left(e^{3X} + 2\, e^{X}\right) + p_2\left[\left(\frac{3}{2} + \frac{1}{2\sqrt{t}}J\right) e^{\left(2-\frac{1}{\sqrt{t}}J\right)X} + \left(\frac{3}{2} - \frac{1}{2\sqrt{t}}J\right) e^{\left(2+\frac{1}{\sqrt{t}}J\right)X}\right],$$

$$Y'(X) = p_1\left(3e^{3X} + 2\, e^{X}\right) + p_2\left[\left(2 - \frac{1}{\sqrt{t}}J\right)\left(\frac{3}{2} + \frac{1}{2\sqrt{t}}J\right) e^{\left(2-\frac{1}{\sqrt{t}}J\right)X} + \left(2 + \frac{1}{\sqrt{t}}J\right)\left(\frac{3}{2} - \frac{1}{2\sqrt{t}}J\right) e^{\left(2+\frac{1}{\sqrt{t}}J\right)X}\right].$$

Substituting the initial conditions: $Y(0) = 1$ in the general solution Y, and $Y'(0) = 1$ in the first derivative of the general solution Y', we get

$$\begin{cases} 3 = p_1(1+2) + p_2\left(\frac{3}{2} + \frac{1}{2\sqrt{t}}J + \frac{3}{2} - \frac{1}{2\sqrt{t}}J\right), \\ 5 = p_1(3+2) + p_2\left[\left(2 - \frac{1}{\sqrt{t}}J\right)\left(\frac{3}{2} + \frac{1}{2\sqrt{t}}J\right) + \left(2 + \frac{1}{\sqrt{t}}J\right)\left(\frac{3}{2} - \frac{1}{2\sqrt{t}}J\right)\right], \end{cases}$$

$$\Rightarrow \begin{cases} 3 = 3p_1 + 3p_2 \\ 5 = 5p_1 + p_2\left[3 - \frac{1}{\sqrt{t}}J - \frac{3}{2\sqrt{t}}J - \frac{1}{2} + 3 - \frac{1}{\sqrt{t}}J + \frac{3}{2\sqrt{t}}J - \frac{1}{2}\right] \end{cases}$$
$$\Rightarrow \begin{cases} 1 = p_1 + p_2 \\ 5 = 5p_1 + \left(5 - \frac{2}{\sqrt{t}}J\right)p_2. \end{cases}$$

Solving these two equations [12], we find $p_1 = 1, p_2 = 0$ and the particular solution of *Eq. (17)* is

$$Y(X) = e^{3X} + 2\, e^{X}.$$

5 | Conclusion

In this paper, we have defined the second-order WFC-DEs. We have focused on solving the second-order constant-coefficient linear homogeneous WFC DE using its characteristic equation, where we have found its general solution. We have solved the related second-order constant-coefficient linear homogeneous IVPs by adding initial conditions. We have explained all examples for this type of DEs and IVPs with WFC constant coefficients and real constant coefficients.

Author Contribution

Conceptualization and Methodology, L. R., writing-reviewing and editing L. R., S. M. and M. A. All authors have read and agreed to the published version of the manuscript.

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Data Availability

All data supporting the reported findings in this research paper are provided within the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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