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Soft Intersection-difference Product of Groups

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Abstract

Soft set theory constitutes a comprehensive mathematical apparatus for modeling and managing uncertainty. Central to this theory are soft set operations and product constructions, which facilitate novel methodologies for addressing problems characterized by parametric data. In the present study, we propose a new product structure for soft sets whose parameter sets possess a group structure, termed the soft intersection-difference product. A rigorous investigation of its fundamental algebraic properties is conducted, encompassing various soft subsets and notions of equality. The findings are anticipated to stimulate further scholarly inquiry, potentially laying the groundwork for a nascent soft group theory derived from this construction. Given that the development of soft algebraic structures fundamentally relies on well-defined soft set operations and products, the study offers a substantial contribution to the theoretical advancement of soft set theory.

Keywords: Soft sets, Soft subsets, Soft equalities, Soft intersection-difference product.

1 | Introduction

Numerous researchers have proposed a variety of mathematical frameworks aimed at modeling and addressing complex problems characterized by uncertainty, vagueness, and ambiguity in domains such as engineering, economics, social sciences, and healthcare. Molodtsov [1] identified inherent limitations in existing frameworks. For instance, fuzzy set theory [2] often encounters challenges in the appropriate specification of membership functions, while probability theory relies on extensive trials to establish the existence of a mean value.

To overcome these limitations, Molodtsov [1] introduced soft set theory as a novel mathematical paradigm, demonstrating its potential applicability in diverse areas such as probability theory, game theory, and operations research. In contrast to classical approaches, soft set theory provides a more adaptable framework

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by eliminating rigid requirements related to approximate descriptions. The seminal work by Maji et al. [3], which applied soft sets to decision-making, laid the groundwork for subsequent advancements. Building upon this foundation, numerous studies [4–10] introduced enhanced decision-making methodologies grounded in soft set theory. Notably, Çağman and Enginoğlu [11] proposed a soft set-based decision-making model, and further introduced the concept of soft matrices [12], formulating decision procedures based on their AND, OR, AND-NOT, and OR-NOT operations. These formulations proved effective in resolving real-world problems under uncertainty.

The adaptability and utility of soft set theory have led to its widespread adoption in decision-making contexts [13-24], where significant developments have emerged. These include bijective and exclusive disjunctive soft sets, generalized uni-int frameworks, soft approximations, operator-based decision processes, reduced and cardinality-inverse soft matrices, soft semantics, and a range of mean and generalized operators on fuzzy soft matrices, as well as soft set-valued mappings.

In recent years, scholarly interest has increasingly turned to the theoretical underpinnings of soft set theory. Maji et al. [25] conducted a foundational analysis involving concepts such as soft subsets and supersets, soft set equality, and basic operations including union, intersection, AND-product, and OR-product. Pei and Miao [26] refined these notions by exploring connections between soft sets and information systems and redefining intersection and subset relations. Further contributions by Ali et al. [27] introduced new operations such as restricted union, restricted intersection, restricted difference, and extended intersection. Subsequent investigations [28-40] have focused on the algebraic structure of soft set operations, correcting earlier conceptual inconsistencies and proposing new methodologies.

Significant progress has been made in the formalization of soft set operations, as evidenced by a wide array of newly defined and rigorously analyzed operations [41–48]. Central to the theory are soft equal relations and soft subsets. Maji et al. [25] initiated the formal definition of soft subsets, which was later extended by Pei and Miao [26] and Feng et al. [29]. Qin and Hong [49] introduced new notions of soft congruence and equality. To generalize Maji's distributive laws, Jun and Yang [50] incorporated broader soft subset classes and proposed J-soft equal relations. Inspired by their work, Liu et al. [51] explored soft L-subsets and soft L-equal relations, revealing that distributive laws do not universally apply across all soft equalities.

Building on this foundation, Feng et al. [52] investigated classifications of soft subsets and the properties of soft product operations introduced in [24], such as the AND- and OR-products, within the framework of soft L-subsets. Their comprehensive analysis addressed commutativity, associativity, and distributivity, resolving previously incomplete findings and demonstrating that soft L-equal relations constitute congruence relations in free soft algebras, where resulting quotient structures form commutative semigroups. For further developments in the theory of soft equalities—including generalized soft equality, soft lattices, relaxed parameter constraints, g-soft and gf-soft equality, and T-soft equality—see [53–57].

Çağman and Enginoğlu [11] revisited the definition of soft set by Maji et al. [25] and its operations to enhance practical applicability. They introduced four distinct product operations: AND-, OR-, AND-NOT-, and OR-NOT-products, alongside the uni-int decision function. These innovations were integrated into a unified decision-making framework, demonstrated through practical applications involving uncertainty. Sezgin et al. [58] subsequently analyzed the AND-product, a pivotal operation in soft set-based decision-making, within various equality frameworks, such as soft L-equality and J-equality. Their work systematically examined its algebraic properties, including idempotency, commutativity, and associativity, comparing these with results involving soft F-subsets, M-equality, L-equality, and J-equality.

The concept of the soft union product was first introduced for rings [59], semigroups [60], and groups [61], forming the basis for the development of soft union ring, semigroup, and group theories. Similarly, the soft intersection product was defined for groups [62], semigroups [63], and rings [64], with corresponding algebraic theories subsequently developed. Due to inherent differences among these algebraic structures, the definitions and properties of these products exhibit structural variations. In particular, the presence of a unit element and inverses in groups imparts unique characteristics to the group-based definitions.

In this study, we propose a novel product for soft sets whose parameter sets form a group structure, referred to as the soft intersection-difference product, constructed within the definitional framework of Çağman and Enginoğlu [11]. A detailed analysis of its algebraic properties is undertaken, considering various soft subsets and equality relations, with the aim of inspiring the development of a new soft group theory rooted in this construct. The remainder of the paper is organized as follows: Section 2 revisits essential concepts in soft set theory; Section 3 introduces the soft intersection-difference product and presents a comprehensive algebraic analysis in relation to different types of soft subsets and equalities. The concluding section offers a summary of the results and outlines directions for future research.

2 | Preliminaries

This section is devoted to revisiting a selection of foundational definitions and structural properties that serve as a theoretical basis for the developments presented in the subsequent section. Although the notion of the soft set was initially proposed by Molodtsov [1], the conceptual framework, including key definitions and operational structures, was later substantially revised by Çağman and Enginoğlu [11] to enhance both theoretical rigor and applicability. Accordingly, the present study is grounded in this refined formulation, which will be employed consistently throughout the paper.

Definition 1 ([11]). Let E be a parameter set, U be a universal set, $P(U)$ be the power set of U, and $\Upsilon \subseteq E$. Then, the soft set $\int_{\Upsilon}$ over U is a function such that $\int_{\Upsilon}: E \to P(U)$, where for all $k \notin \Upsilon$, $\int_{\Upsilon}(k) = \emptyset$. That is,

$$\int_{\Upsilon} = \{(k, \int_{\Upsilon}(k)): k \in E\}.$$

From now on, soft set over U is abbreviated by SS.

Definition 2 ([11]). Let $\int_{\mathcal{H}}$ be an SS. If $\int_{\mathcal{H}}(k) = \emptyset$, for all $k \in E$, then $\int_{\mathcal{H}}$ is called a null SS and indicated by $\emptyset_E$, and if $\int_{\mathcal{H}}(k) = U$, for all $k \in E$, then $\int_{\mathcal{H}}$ is called an absolute SS and indicated by U_E.

Definition 3 ([11]). Let $\int_{\mathcal{H}}, \int_{\aleph}$ be two SSs. If $\int_{\mathcal{H}}(k) \subseteq \int_{\aleph}(k)$, for all $k \in E$, then $\int_{\mathcal{H}}$ is a soft subset of $\int_{\aleph}$ and indicated by $\int_{\mathcal{H}} \tilde{\subseteq} \int_{\aleph}$. If $\int_{\mathcal{H}}(k) = \int_{\aleph}(k)$, for all $k \in E$, then $\int_{\mathcal{H}}$ is called soft equal to $\int_{\aleph}$, and denoted by $\int_{\mathcal{H}} = \int_{\aleph}$.

Definition 4 ([11]). Let $\int_{\mathcal{H}}, \int_{\aleph}$ be two SSs. The union of $\int_{\mathcal{H}}$ and $\int_{\aleph}$ is the SS $\int_{\mathcal{H}} \tilde{\cup} \int_{\aleph}$, where $(\int_{\mathcal{H}} \tilde{\cup} \int_{\aleph})(w) = \int_{\mathcal{H}}(w) \cup \int_{\aleph}(w)$, for all $w \in E$.

Definition 5 ([11]). Let $\int_{\mathcal{H}}$ be an SS. Then, the complement of $\int_{\mathcal{H}}$, denoted by $(\int_{\mathcal{H}})^c$, is defined by the SS $\int_{\mathcal{H}}{}^c: E \to P(U)$ such that $\int_{\mathcal{H}}{}^c(e) = U \setminus \int_{\mathcal{H}}(e) = (\int_{\mathcal{H}}(e))'$, for all $e \in E$.

Definition 6 ([65]). Let $\int_K$ and $\mathfrak{q}_L$ be two SSs. Then, $\int_K$ is called a soft S-subset of $\mathfrak{q}_L$, denoted by $\int_K \tilde{\subseteq}_S \mathfrak{q}_L$, if for all $e \in E$, $\int_K(e) = \bar{A}$ and $\mathfrak{q}_K(e) = B$, where $\bar{A}$ and B are two fixed sets and $\bar{A} \subseteq B$. Moreover, two SSs $\int_K$ and $\mathfrak{q}_L$ are said to be soft S-equal, denoted by $\int_K =_S \mathfrak{q}_L$, if $\int_K \tilde{\subseteq}_S \mathfrak{q}_L$ and $\mathfrak{q}_L \tilde{\subseteq}_S \int_K$.

It is obvious that if $\int_K =_S \mathfrak{q}_L$, then $\int_K$ and $\mathfrak{q}_L$ are the same constant functions, that is, for all $e \in E$, $\int_K(e) = \mathfrak{q}_K(e) = \bar{A}$, where $\bar{A}$ is a fixed set.

Definition 7 (65]). Let $\int_K$ and $\mathfrak{q}_L$ be two SSs. Then, $\int_K$ is called a soft A-subset of $\mathfrak{q}_L$, denoted by $\int_K \tilde{\subseteq}_A \mathfrak{q}_L$, if for each $g, h \in E$, $\int_K(g) \subseteq \mathfrak{q}_L(h)$.

Definition 8 ([65]). Let $\int_K$ and $\mathfrak{q}_L$ be two SSs. Then, $\int_K$ is called a soft S-complement of $\mathfrak{q}_L$, denoted by $\int_K =_S (\mathfrak{q}_L)^c$, if for all $e \in E$, $\int_K(e) = \bar{A}$ and $\mathfrak{q}_L(e) = B$, where $\bar{A}$ and B are two fixed sets and $\bar{A} = B'$. Here, $B' = U \setminus B$.

For additional information on SSs, we refer to [66-82].

Definition 6 ([61]). Let $\int_G$ and $\mathfrak{q}_G$ be two SSs, where G is a group. Then, the soft intersection-union product $\int_G \otimes_{i/u} \mathfrak{q}_G$ is defined by

$$(ʆ_G \otimes_{i/u} ɖ_G)(ƙ) = \bigcap_{ƙ=łɯ} (ʆ_G(ł) \cup ɖ_G(ɯ)), \quad ł, ɯ \in G,$$

for all $ƙ \in G$.

Definition 7 ([83]). Let $ʆ_G$ and $ɖ_G$ be two SSs where G is a group. Then, the soft intersection-symmetric difference product $ʆ_G \otimes_{i/s} ɖ_G$ is defined by

$$(ʆ_G \otimes_{i/s} ɖ_G)(ƙ) = \bigcap_{ƙ=łɯ} (ʆ_G(ł) \Delta ɖ_G(ɯ)), \quad ł, ɯ \in G,$$

for all $ƙ \in G$.

Definition 8 ([83]). Let $ʆ_G$ and $ɖ_G$ be two SSs, where G is a group. Then, the soft union-difference product $ʆ_G \otimes_{u/d} ɖ_G$ is defined by

$$(ʆ_G \otimes_{u/d} ɖ_G)(ƙ) = \bigcup_{ƙ=łɯ} (ʆ_G(ł) \setminus ɖ_G(ɯ)), \quad ł, ɯ \in G,$$

for all $ƙ \in G$.

3 | Soft Intersection-Difference Product of Groups

In this section, we introduce a novel binary operation for soft sets, referred to as the soft intersection-difference product, defined in the context of soft sets whose parameter set is a group structure. A rigorous algebraic analysis of this product is carried out, with particular attention devoted to its characteristics under diverse notions of soft equalities and various classifications of soft subsets. Theoretical findings are further elucidated through representative examples that illustrate the structural characteristics of the proposed operation.

From now on, G denotes a group, $S_G(U)$ is the collections of SSs, whose parameter set is G, and all the SSs are the elements of $S_G(U)$.

Definition 12. Let $ʆ_G$ and $ɖ_G$ be two SSs. Then, the soft intersection-difference product $ʆ_G \otimes_{i/d} ɖ_G$ is defined by

$$(ʆ_G \otimes_{i/d} ɖ_G)(ƙ) = \bigcap_{ƙ=łɯ} (ʆ_G(ł) \setminus ɖ_G(ɯ)), \quad ł, ɯ \in G,$$

for all $ƙ \in G$.

Note here that since G is a group, there always exists $ł, ɯ \in G$ such that $ƙ = łɯ$, for all $ƙ \in G$. Let the order of the group be n, that is, $|G| = n$. Then, it is obvious that there exist n distint representations for each $ƙ \in G$ such that $ƙ = łɯ$, where $ł, ɯ \in G$.

Note 1: The soft intersection-difference product is well-defined in $S_G(U)$. In fact, let $ʆ_G, ɖ_G, ƀ_G, ζ_G \in S_G(U)$ such that $(ʆ_G, ɖ_G) = (ƀ_G, ζ_G)$. Then, $ʆ_G = ƀ_G$ and $ɖ_G = ζ_G$, implying that $ʆ_G(ƙ) = ƀ_G(ƙ)$ and $ɖ_G(ƙ) = ζ_G(ƙ)$, for all $ƙ \in G$. Thereby, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɖ_G)(ƙ) = \bigcap_{ƙ=łɯ} (ʆ_G(ł) \setminus ɖ_G(ɯ)) = \bigcap_{ƙ=łɯ} (ƀ_G(ł) \setminus ζ_G(ɯ)) = (ƀ_G \otimes_{i/d} ζ_G)(ƙ)$$

Hence, $ʆ_G \otimes_{i/d} ɖ_G = ƀ_G \otimes_{i/d} ζ_G$.

Example 1. Consider the group $G = \{Ձ, ɓ\}$ with the following operation:

$\cdot$	Ձ	ɓ
Ձ	Ձ	ɓ
ɓ	ɓ	Ձ

Let $ʆ_G$ and $ɋ_G$ be two SSs over $U = D_2 = \{<x,y>: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:

$$ʆ_G = \{(Ձ, \{e, x, yx\}), (ɓ, \{x, yx\})\},\ ɋ_G = \{(Ձ, \{e, y, yx\}), (ɓ, \{e, y\})\}.$$

Since $Ձ = ՁՁ = ɓɓ$, $(ʆ_G \otimes_{i/d} ɋ_G)(Ձ) = (ʆ_G(Ձ) \setminus ɋ_G(Ձ)) \cap (ʆ_G(ɓ) \setminus ɋ_G(ɓ)) = \{x\}$, and since $ɓ = Ձɓ = ɓՁ$, $(ʆ_G \otimes_{i/d} ɋ_G)(ɓ) = (ʆ_G(Ձ) \setminus ɋ_G(ɓ)) \cap (ʆ_G(ɓ) \setminus ɋ_G(Ձ)) = \{x\}$ is obtained. Hence,

$$ʆ_G \otimes_{i/d} ɋ_G = \{(Ձ, \{x\}), (ɓ, \{x\})\}.$$

Proposition 1. The set $S_G(U)$ is closed under the soft intersection-difference product. That is, if $ʆ_G$ and $ɋ_G$ are two SSs, then so is $ʆ_G \otimes_{i/d} ɋ_G$.

Proof: It is obvious that the soft intersection-difference product is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the soft intersection-difference product.

Proposition 2. The soft intersection-difference product is not associative in $S_G(U)$.

Proof: Consider the group G and the SSs $ʆ_G$ and $ɋ_G$ in *Example 1*. Let $ɦ_G$ be an SS over $U = \{e, x, y, yx\}$ such that $ɦ_G = \{(Ձ, \{e, y\}), (ɓ, \{y, yx\})\}$.

Since $ʆ_G \otimes_{i/d} ɋ_G = \{(Ձ, \{x\}), (ɓ, \{x\})\}$, then

$$(ʆ_G \otimes_{i/d} ɋ_G) \otimes_{i/d} ɦ_G = \{(Ձ, \{x\}), (ɓ, \{x\})\}.$$

Moreover, since $ɋ_G \otimes_{i/d} ɦ_G = \{(Ձ, \emptyset), (ɓ, \emptyset)\}$, then

$$ʆ_G \otimes_{i/d} (ɋ_G \otimes_{i/d} ɦ_G) = \{(Ձ, \{x, yx\}), (ɓ, \{x, yx\})\}.$$

Thereby, $(ʆ_G \otimes_{i/d} ɋ_G) \otimes_{i/d} ɦ_G \neq ʆ_G \otimes_{i/d} (ɋ_G \otimes_{i/d} ɦ_G)$.

Proposition 3. The soft intersection-difference product is not commutative in $S_G(U)$.

Proof: Consider the SSs $ʆ_G$ and $ɋ_G$ in *Example 1*. Then,

$$ʆ_G \otimes_{i/d} ɋ_G = \{(Ձ, \{x\}), (ɓ, \{x\})\},$$

and

$$ɋ_G \otimes_{i/d} ʆ_G = \{(Ձ, \{y\}), (ɓ, \{y\})\},$$

implying that $ʆ_G \otimes_{i/d} ɋ_G \neq ɋ_G \otimes_{i/d} ʆ_G$.

Proposition 4. The soft intersection-difference product is not idempotent in $S_G(U)$.

Proof: Consider the SS $ʆ_G = \{(Ձ, \{e, x, yx\}), (ɓ, \{x, yx\})\}$ in *Example 1*. Then,

$$ʆ_G \otimes_{i/d} ʆ_G = \{(Ձ, \emptyset), (ɓ, \emptyset)\},$$

implying that $ʆ_G \otimes_{i/d} ʆ_G \neq ʆ_G$.

Proposition 5. Let $ʆ_G$ be a constant SS. Then, $ʆ_G \otimes_{i/d} ʆ_G = \emptyset_G$.

Proof: Let $ʆ_G$ be a constant SS such that, for all $ƙ \in G$, $ʆ_G(ƙ) = \bar{A}$, where $\bar{A}$ is a fixed set. Then, for all $ƙ \in G$,

Thereby, $ʆ_G \otimes_{i/d} ʆ_G = \emptyset_G$.

$$(\int_G \otimes_{i/d} \int_G)(k) = \bigcap_{k=tw} (\int_G(t) \setminus \int_G(w)) = \emptyset_G(k).$$

Remark 1. Let $S_G{}^*(U)$ be the collection of all constant SSs. Then, the soft intersection-difference product is not idempotent in $S_G{}^*(U)$ either.

Proposition 6. $\emptyset_G$ is the left absorbing element of the soft intersection-difference product in $S_G(U)$.

Proof: Let $x \in G$. Then,

$$(\emptyset_G \otimes_{i/d} \int_G)(k) = \bigcap_{k=tw} (\emptyset_G(t) \setminus \int_G(w)) = \bigcap_{k=tw} (\emptyset \setminus \int_G(w)) = \emptyset_G(k).$$

Thus, $\emptyset_G \otimes_{i/d} \int_G = \emptyset_G$.

Proposition 7. $\emptyset_G$ is not the right absorbing element of the soft intersection-difference product in $S_G(U)$.

Proof: Consider the SS $\int_G = \{(a, \{e, x, yx\}), (b, \{x, yx\})\}$ in *Example 1*. Then,

$$\int_G \otimes_{i/d} \emptyset_G = \{(a, \{x, yx\}), (b, \{x, yx\})\},$$

implying that $\int_G \otimes_{i/d} \emptyset_G \neq \emptyset_G$.

Remark 2. $\emptyset_G$ is not the absorbing element of soft intersection-difference product in $S_G(U)$.

Proposition 8. Let $\int_G$ be a constant SS. Then, $\int_G \otimes_{i/d} \emptyset_G = \int_G$.

Proof: Let $\int_G$ be a constant SS such that, for all $k \in G$, $\int_G(k) = \bar{A}$, where $\bar{A}$ is a fixed set. Then, for all $k \in G$,

$$(\int_G \otimes_{i/d} \emptyset_G)(k) = \bigcap_{k=tw} (\int_G(t) \setminus \emptyset_G(w)) = \int_G(k).$$

Thereby, $\int_G \otimes_{i/d} \emptyset_G = \int_G$.

Remark 3. $\emptyset_G$ is not the absorbing element of soft intersection-difference product in $S_G{}^*(U)$ either. Moreover, $\emptyset_G$ is the right identity element of soft intersection-difference product in $S_G{}^*(U)$.

Proposition 9. Let $\int_G$ be an SS. Then, $\int_G \otimes_{i/d} U_G = \emptyset_G$.

Proof: Let $\int_G$ be an SS. Then, for all $k \in G$,

$$(\int_G \otimes_{i/d} U_G)(k) = \bigcap_{k=tw} (\int_G(t) \setminus U_G(w)) = \emptyset_G(k).$$

Thereby, $\int_G \otimes_{i/d} U_G = \emptyset_G$.

Proposition 10. Let $\int_G$ be a constant SS. Then, $U_G \otimes_{i/d} \int_G = \int_G{}^c$.

Proof: Let $\int_G$ be a constant SS such that, for all $k \in G$, $\int_G(k) = \bar{A}$, where $\bar{A}$ is a fixed set. Then, for all $k \in G$,

$$(\int_G \otimes_{i/d} U_G)(k) = \bigcap_{k=tw} (U_G(t) \setminus \int_G(w)) = \int_G{}^c(k).$$

Thereby, $U_G \otimes_{i/d} \int_G = \int_G{}^c$.

Proposition 11. Let $\int_G$ be a constant SS. Then, $\int_G{}^c \otimes_{i/d} \int_G = \int_G{}^c$.

Proof: Let $\int_G$ be a constant SS such that, for all $k \in G$, $\int_G(k) = \bar{A}$, where $\bar{A}$ is a fixed set. Then, for all $k \in G$,

$$(\int_G{}^c \otimes_{i/d} \int_G)(k) = \bigcap_{k=tw} (\int_G{}^c(t) \setminus \int_G(w)) = \int_G{}^c(k).$$

Thereby, $\int_G{}^c \otimes_{i/d} \int_G = \int_G{}^c$.

Proposition 12. Let $ʆ_G$ be a constant SS. Then, $ʆ_G \otimes_{i/d} {ʆ_G}^c = ʆ_G$.

Proof: Let $ʆ_G$ be a constant SS such that, for all $ƙ \in G$, $ʆ_G(ƙ) = \bar{A}$, where $\bar{A}$ is a fixed set. Then, for all $ƙ \in G$,

$$\left(ʆ_G \otimes_{i/d} {ʆ_G}^c\right)(ƙ) = \bigcap_{ƙ=ŧɯ} \left(ʆ_G(ŧ) \setminus {ʆ_G}^c(ɯ)\right) = ʆ_G(ƙ).$$

Thereby, $ʆ_G \otimes_{i/d} {ʆ_G}^c = ʆ_G$.

Proposition 13. Let $ʆ_G$, $ɋ_G$ and $ɧ_G$ be three SSs. If $ʆ_G \cong ɋ_G$, then $ʆ_G \otimes_{i/d} ɧ_G \cong ɋ_G \otimes_{i/d} ɧ_G$ and $ɧ_G \otimes_{i/d} ɋ_G \cong ɧ_G \otimes_{i/d} ʆ_G$.

Proof: Let $ʆ_G$, $ɋ_G$ and $ɧ_G$ be three SSs such that $ʆ_G \cong ɋ_G$. Then, for all $ƙ \in G$, $ʆ_G(ƙ) \subseteq ɋ_G(ƙ)$ and $(ɋ_G(ƙ))' \subseteq (ʆ_G(ƙ))'$. Thus, for all $ƙ \in G$,

$$\left(ʆ_G \otimes_{i/d} ɧ_G\right)(ƙ) = \bigcap_{ƙ=ŧɯ} \left(ʆ_G(ŧ) \setminus ɧ_G(ɯ)\right) \subseteq \bigcap_{ƙ=ŧɯ} \left(ɋ_G(ŧ) \setminus ɧ_G(ɯ)\right) = \left(ɋ_G \otimes_{i/d} ɧ_G\right)(ƙ),$$

for all $ƙ \in G$, implying that $ʆ_G \otimes_{i/d} ɧ_G \cong ɋ_G \otimes_{i/d} ɧ_G$. Similarly, for all $ƙ \in G$,

$$\left(ɧ_G \otimes_{i/d} ɋ_G\right)(ƙ) = \bigcap_{ƙ=ŧɯ} \left(ɧ_G(ŧ) \setminus ɋ_G(ɯ)\right) \subseteq \bigcap_{ƙ=ŧɯ} \left(ɧ_G(ŧ) \setminus ʆ_G(ɯ)\right) = \left(ɧ_G \otimes_{i/d} ʆ_G\right)(ƙ),$$

for all $ƙ \in G$, implying that $ɧ_G \otimes_{i/d} ɋ_G \cong ɧ_G \otimes_{i/d} ʆ_G$.□

Proposition 14. Let $ʆ_G$, $ɋ_G$, $ƀ_G$, and $ζ_G$ be four SSs. If $ƀ_G \cong ʆ_G$ and $ζ_G \cong ɋ_G$, then $ζ_G \otimes_{i/d} ʆ_G \cong ɋ_G \otimes_{i/d} ƀ_G$ and $ƀ_G \otimes_{i/d} ɋ_G \cong ʆ_G \otimes_{i/d} ζ_G$.

Proof: Let $ʆ_G$, $ɋ_G$, $ƀ_G$ and $ζ_G$ be four SSs such that $ƀ_G \cong ʆ_G$ and $ζ_G \cong ɋ_G$. Then, for all $ƙ \in G$, $ƀ_G(ƙ) \subseteq ʆ_G(ƙ)$ and $ζ_G(ƙ) \subseteq ɋ_G(ƙ)$. Thus, $\left(ʆ_G(ƙ)\right)' \subseteq \left(ƀ_G(ƙ)\right)'$ and $\left(ɋ_G(ƙ)\right)' \subseteq \left(ζ_G(ƙ)\right)'$, for all $ƙ \in G$. Hence, for all $ƙ \in G$,

$$\left(ζ_G \otimes_{i/d} ʆ_G\right)(ƙ) = \bigcap_{ƙ=ŧɯ} \left(ζ_G(ŧ) \setminus ʆ_G(ɯ)\right) \subseteq \bigcap_{ƙ=ŧɯ} \left(ɋ_G(ŧ) \setminus ƀ_G(ɯ)\right) = \left(ɋ_G \otimes_{i/d} ƀ_G\right)(ƙ),$$

implying that $ζ_G \otimes_{i/d} ƀ_G \cong ɋ_G \otimes_{i/d} ʆ_G$. Similarly, for all $ƙ \in G$,

$$\left(ƀ_G \otimes_{i/d} ɋ_G\right)(ƙ) = \bigcap_{ƙ=ŧɯ} \left(ƀ_G(ŧ) \setminus ɋ_G(ɯ)\right) \subseteq \bigcap_{ƙ=ŧɯ} \left(ʆ_G(ŧ) \setminus ζ_G(ɯ)\right) = \left(ʆ_G \otimes_{i/d} ζ_G\right)(ƙ),$$

is obtained. Thereby, $ƀ_G \otimes_{i/d} ɋ_G \cong ʆ_G \otimes_{i/d} ζ_G$.□

Theorem 1. Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, $ʆ_G \otimes_{i/d} ɋ_G = U_G$ if and only if $ʆ_G = U_G$ and $ɋ_G = \emptyset_G$.

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs. Suppose that $ʆ_G = U_G$ and $ɋ_G = \emptyset_G$. Hence, for all $ƙ \in G$, $ʆ_G(ƙ) = U_G(ƙ) = U$ and $ɋ_G(ƙ) = \emptyset_G(ƙ) = \emptyset$. Thus, for all $ƙ \in G$,

$$\left(ʆ_G \otimes_{i/d} ɋ_G\right)(ƙ) = \bigcap_{ƙ=ŧɯ} \left(ʆ_G(ŧ) \setminus ɋ_G(ɯ)\right) = \bigcap_{ƙ=ŧɯ} \left(U_G(ŧ) \setminus \emptyset_G(ɯ)\right) = U_G(ƙ).$$

Thereby, $ʆ_G \otimes_{i/d} ɋ_G = U_G$.

Conversely, suppose that $ʆ_G \otimes_{i/d} ɋ_G = U_G$. Then, $\left(ʆ_G \otimes_{i/d} ɋ_G\right)(ƙ) = U_G(ƙ) = U$, for all $ƙ \in G$. Thus, for all $ƙ \in G$,

$$U_G(ƙ) = U = ʆ_G \otimes_{i/d} ɋ_G(ƙ) = \bigcap_{ƙ=ƚɯ} (ʆ_G(ƚ) \setminus ɋ_G(ɯ)).$$

This implies that $ʆ_G(ƚ) \setminus ɋ_G(ɯ) = U$, for all $ƚ, ɯ \in G$. Thus, $ʆ_G(ƚ) = U$ and $ɋ_G(ɯ) = \emptyset$, for all $ƚ, ɯ \in G$. Thereby, $ʆ_G = U_G$ and $ɋ_G = \emptyset_G$.

Proposition 15. Let $ʆ_G$ and $ɋ_G$ be two SSs. If $ʆ_G \widetilde{\subseteq}_A ɋ_G$, then $ʆ_G \otimes_{i/d} ɋ_G = \emptyset_G$.

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs and $ʆ_G \widetilde{\subseteq}_A ɋ_G$. Thus, $ʆ_G(Ձ) \subseteq ɋ_G(ɓ)$ for each $Ձ, ɓ \in G$. Hence, for all $ƙ \in G$,

$$ʆ_G \otimes_{i/d} ɋ_G(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = \emptyset = \emptyset_G(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G = \emptyset_G$ is obtained.

Proposition 16. Let $ʆ_G$ and $ɋ_G$ be two SSs. If $ʆ_G \widetilde{\subseteq}_S (ɋ_G)^c$, then $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G$.

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs and $ʆ_G \widetilde{\subseteq}_S (ɋ_G)^c$. Hence, for all $Ձ \in G$, $ʆ_G(Ձ) = \bar{A}$ and $ɋ_G(Ձ) = B$, where $\bar{A}$ and B are two fixed sets and $\bar{A} \subseteq B'$. Thus, for all $ƙ \in G$,

$$ʆ_G \otimes_{i/d} ɋ_G(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \cap {ɋ_G}^c(ɯ)\right) = ʆ_G(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G$.

Proposition 17. Let $ʆ_G$ and $ɋ_G$ be two SSs. If $({ɋ_G}^c) \widetilde{\subseteq}_S ʆ_G$, then $ʆ_G \otimes_{i/d} ɋ_G = {ɋ_G}^c$.

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs and ${ɋ_G}^c \widetilde{\subseteq}_S ʆ_G$. Hence, for all $Ձ \in G$, $ʆ_G(Ձ) = \bar{A}$ and $ɋ_G(Ձ) = B$, where $\bar{A}$ and B are two fixed sets and $B' \subseteq \bar{A}$. Thus, for all $ƙ \in G$,

$$ʆ_G \otimes_{i/d} ɋ_G(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \cap {ɋ_G}^c(ɯ)\right) = {ɋ_G}^c(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G = {ɋ_G}^c$.

Proposition 18. Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, $\left(ʆ_G \otimes_{i/d} ɋ_G\right)^c = ʆ_G \otimes_{u/p} ɋ_G$.

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, for all $ƙ \in G$,

$$\left(ʆ_G \otimes_{i/d} ɋ_G\right)^c(ƙ) = \left(\bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right)\right)' = \bigcup_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right)'$$

$$= \bigcup_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \cap {ɋ_G}^c(ɯ)\right)' = \bigcup_{ƙ=ƚɯ} \left({ʆ_G}^c(ƚ) \cup ɋ_G(ɯ)\right) = \left(ʆ_G \otimes_{u/p} ɋ_G\right)(ƙ).$$

Thereby, $\left(ʆ_G \otimes_{i/d} ɋ_G\right)^c = ʆ_G \otimes_{u/p} ɋ_G$.

For more on soft union-plus product of groups, we refer to [84].

Proposition 19. Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, $ʆ_G \otimes_{i/d} ɋ_G \widetilde{\subseteq} ʆ_G \otimes_{i/u} ɋ_G$.

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, for all $ƙ \in G$,

$$\left(ʆ_G \otimes_{i/d} ɋ_G\right)(ƙ) = \bigcap_{ƙ=ƚɯ} (ʆ_G(ƚ) \setminus ɋ_G(ɯ)) \subseteq \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \cup ɋ_G(ɯ)\right) = \left(ʆ_G \otimes_{i/u} ɋ_G\right)(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G \widetilde{\subseteq} ʆ_G \otimes_{i/u} ɋ_G$. □

Proposition 20. Let $ʆ_G$ and $ɋ_G$ be two SSs. If one of the assertions following is satisfied, then $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{i/u} ɋ_G$:

I. $ɋ_G = \emptyset_G$

II. $ʆ_G = \emptyset_G$ and $ɋ_G(ƙ) \cap ɋ_G(ŧ) = \emptyset$ for all $ƙ, ŧ \in G$

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs

I. Let $ɋ_G = \emptyset_G$.Then, for all $ƙ \in G$, $ɋ_G(ƙ) = \emptyset_G(ƙ) = \emptyset$. Thus, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \setminus ɋ_G(ɯ)\big) = \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \setminus \emptyset_G(ɯ)\big)$$
$$= \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \cup \emptyset_G(ɯ)\big) = \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \cup ɋ_G(ɯ)\big) = (ʆ_G \otimes_{i/u} ɋ_G)(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{i/u} ɋ_G$.

II. Let $ʆ_G = \emptyset_G$ and $ɋ_G(ƙ) \cap ɋ_G(ŧ) = \emptyset$, for all $ƙ, ŧ \in G$. Then, for all $ƙ \in G$, $ʆ_G(ƙ) = \emptyset_G(ƙ) = \emptyset$. Hence, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \setminus ɋ_G(ɯ)\big) = \bigcap_{ƙ=ŧɯ} \big(\emptyset_G(ŧ) \setminus ɋ_G(ɯ)\big)$$
$$= \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \cup ɋ_G(ɯ)\big) = \emptyset = (ʆ_G \otimes_{i/u} ɋ_G)(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{i/u} ɋ_G$.

Proposition 21. Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, $ʆ_G \otimes_{i/d} ɋ_G \,\tilde{\subseteq}\, ʆ_G \otimes_{i/s} ɋ_G$.

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \setminus ɋ_G(ɯ)\big) \subseteq \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \Delta ɋ_G(ɯ)\big) = (ʆ_G \otimes_{i/s} ɋ_G)(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G \,\tilde{\subseteq}\, ʆ_G \otimes_{i/s} ɋ_G$. □

Proposition 22. Let $ʆ_G$ and $ɋ_G$ be two SSs. If one of the assertions following is satisfied, then $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{i/s} ɋ_G$:

I. $ɋ_G \,\tilde{\subseteq}_S\, ʆ_G$

II. $ʆ_G =_S ɋ_G$

III. $ʆ_G = \emptyset_G$ and $ɋ_G(ƙ) \cap ɋ_G(ŧ) = \emptyset$ for all $ƙ, ŧ \in G$

IV. $ɋ_G = \emptyset_G$

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs.

I. Suppose that $ɋ_G \,\tilde{\subseteq}_S\, ʆ_G$. Hence, for all $Ձ \in G$, $ɋ_G(Ձ) = \bar{A}$ and $ʆ_G(Ձ) = B$, where $\bar{A}$ and B are two fixed sets and $\bar{A} \subseteq B$. Thus, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \setminus ɋ_G(ɯ)\big) = \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \Delta ɋ_G(ɯ)\big) = (ʆ_G \otimes_{i/s} ɋ_G)(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{i/s} ɋ_G$.

II. It follows from *Proposition 22 (I).*

III. Let $ʆ_G = \emptyset_G$ and $ɋ_G(ƙ) \cap ɋ_G(ƚ) = \emptyset$ for all $ƙ, ƚ \in G$. Then, for all $ƙ \in G$, $ʆ_G(ƙ) = \emptyset_G(ƙ) = \emptyset$. Hence, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = \bigcap_{ƙ=ƚɯ} \left(\emptyset_G(ƚ) \setminus ɋ_G(ɯ)\right) = \emptyset_G(ƙ).$$

Similarly, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \Delta ɋ_G(ɯ)\right) = \bigcap_{ƙ=ƚɯ} \left(\emptyset_G(ƚ) \Delta ɋ_G(ɯ)\right) = \emptyset_G(ƙ).$$

Hence, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{i/s} ɋ_G$.

IV. Let $ɋ_G = \emptyset_G$. Then, for all $ƙ \in G$, $ɋ_G(ƙ) = \emptyset_G(ƙ) = \emptyset$. Thus, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus \emptyset_G(ɯ)\right) =$$

$$= \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \Delta ɋ_G(ɯ)\right) = (ʆ_G \otimes_{i/s} ɋ_G)(ƙ).$$

Thereby, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{i/s} ɋ_G$.

Proposition 23. Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, $ʆ_G \otimes_{i/d} ɋ_G \,\tilde{\subseteq}\, ʆ_G \otimes_{u/d} ɋ_G$.

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs. Then, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ(ɯ)\right) \subseteq \bigcup_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ(ɯ)\right) = (ʆ_G \otimes_{u/d} ɋ_G)(ƙ).$$

Thus, $ʆ_G \otimes_{i/d} ɋ_G \,\tilde{\subseteq}\, ʆ_G \otimes_{u/d} ɋ_G$. □

Proposition 24. Let $ʆ_G$ and $ɋ_G$ be two SSs. If one of the assertions following is satisfied, then $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{u/d} ɋ_G$.

I. $ɋ_G \,\tilde{\subseteq}_S\, ʆ_G$

II. $ʆ_G \,\tilde{\subseteq}_A\, ɋ_G$

III. $ʆ_G =_S ɋ_G$

IV. $ʆ_G =_S (ɋ_G)^c$

Proof: Let $ʆ_G$ and $ɋ_G$ be two SSs.

I. Suppose that $ɋ_G \,\tilde{\subseteq}_S\, ʆ_G$. Hence, for all $Ձ \in G$, $ɋ_G(Ձ) = Ā$, $ʆ_G(Ձ) = B$, where $Ā$ and B are two fixed sets and $Ā \subseteq B$. Thus, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = \bigcup_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = (ʆ_G \otimes_{u/d} ɋ_G)(ƙ).$$

Hence, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{u/d} ɋ_G$.

II. Suppose that $ʆ_G \,\tilde{\subseteq}_A\, ɋ_G$. Then, $ʆ_G(Ձ) \subseteq ɋ_G(ɓ)$, for each $Ձ, ɓ \in G$. Thus, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = \emptyset$$

$$= \bigcup_{ƙ=ƚɯ} \left(ʆ_G(ƚ) \setminus ɋ_G(ɯ)\right) = (ʆ_G \otimes_{u/d} ɋ_G)(ƙ).$$

Hence, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{u/d} ɋ_G$.

III. It follows from *Proposition 24 (I)*.

IV. Let $ʆ_G =_S (ɋ_G)^c$. Then, for all $ƙ \in G$, $ʆ_G(ƙ) = \bar{A}$, $ɋ_G(ƙ) = B$, where $\bar{A}$ and B are two fixed sets and $\bar{A} = B'$. Thus, for all $ƙ \in G$,

$$(ʆ_G \otimes_{i/d} ɋ_G)(ƙ) = \bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \setminus ɋ_G(ɯ)\big) = \bigcup_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \setminus ɋ_G(ɯ)\big) = (ʆ_G \otimes_{u/d} ɋ_G)(ƙ).$$

Hence, $ʆ_G \otimes_{i/d} ɋ_G = ʆ_G \otimes_{u/d} ɋ_G$.

Proposition 25. The soft intersection-difference product distributes over soft intersection operation from the right side.

Proof: Let $ʆ_G$, $ɋ_G$ and $ɦ_G$ be three SSs. Then, for all $ƙ \in G$,

$$\begin{aligned}
((ʆ_G \tilde{\cap} ɋ_G) \otimes_{i/d} ɦ_G)(ƙ) &= \bigcap_{ƙ=ŧɯ} \big((ʆ_G \tilde{\cap} ɋ_G)(ŧ) \setminus ɦ_G(ɯ)\big) \\
&= \bigcap_{ƙ=ŧɯ} \Big(\big(ʆ_G(ŧ) \cap ɋ_G(ŧ)\big) \setminus ɦ_G(ɯ)\Big) \\
&= \bigcap_{ƙ=ŧɯ} \big[\big(ʆ_G(ŧ) \setminus ɦ_G(ɯ)\big) \cap \big(ɋ_G(ŧ) \setminus ɦ_G(ɯ)\big)\big] \\
&= \left[\bigcap_{ƙ=ŧɯ} \big(ʆ_G(ŧ) \setminus ɦ_G(ɯ)\big)\right] \cap \left[\bigcap_{ƙ=ŧɯ} \big(ɋ_G(ŧ) \setminus ɦ_G(ɯ)\big)\right] \\
&= (ʆ_G \otimes_{i/d} ɦ_G)(ƙ) \cap (ɋ_G \otimes_{i/d} ɦ_G)(ƙ) \\
&= \big((ʆ_G \otimes_{i/d} ɦ_G) \tilde{\cap} (ɋ_G \otimes_{i/d} ɦ_G)\big)(ƙ).
\end{aligned}$$

Thus, $(ʆ_G \tilde{\cap} ɋ_G) \otimes_{i/d} ɦ_G = (ʆ_G \otimes_{i/d} ɦ_G) \tilde{\cap} (ɋ_G \otimes_{i/d} ɦ_G)$.

Example 2. Consider the SSs $ʆ_G$ and $ɋ_G$ in *Example 1*. Let $ɦ_G$ be an SS as follows: $ɦ_G = \{(Ձ, \{e, yx\}), (Ϭ, \{x, y\})\}$. Since $ʆ_G \otimes_{i/d} ɦ_G = \{(Ձ, \emptyset), (Ϭ, \emptyset)\}$ and $ɋ_G \otimes_{i/d} ɦ_G = \{(Ձ, \emptyset), (Ϭ, \emptyset)\}$, then

$$(ʆ_G \otimes_{i/d} ɦ_G) \tilde{\cap} (ɋ_G \otimes_{i/d} ɦ_G) = \{(Ձ, \emptyset), (Ϭ, \emptyset)\}.$$

Moreover, since $ʆ_G \tilde{\cap} ɋ_G = \{(Ձ, \{e, yx\}), (Ϭ, \emptyset)\}$,

$$(ʆ_G \tilde{\cap} ɋ_G) \otimes_{i/d} ɦ_G = \{(Ձ, \emptyset), (Ϭ, \emptyset)\}.$$

Thus, $(ʆ_G \tilde{\cap} ɋ_G) \otimes_{i/d} ɦ_G = (ʆ_G \otimes_{i/d} ɦ_G) \tilde{\cap} (ɋ_G \otimes_{i/d} ɦ_G)$.□

4 | Conclusion

This study introduces a novel binary operation for soft sets whose parameter sets possess a group structure, termed the soft intersection-difference product. A comprehensive examination of its foundational algebraic properties is undertaken, with particular emphasis on its interaction with various classes of soft subsets and equality relations. The theoretical framework proposed herein not only addresses existing gaps but also offers a potential pathway for the development of a new branch of soft group theory grounded in this product. Prospective research directions may include the formulation of additional soft product operations and a deeper exploration of soft equality structures, both of which are expected to contribute significantly to the advancement of the theoretical and applied dimensions of soft set theory.

Author Contributions

All authors contributed to the study's conception and design. AS supervised the study, performed material preparation. ZA performed data collection, and analysis, and wrote the first draft of the manuscript. All authors read and approved the final manuscript.

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Data Availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author (Aslıhan Sezgin, aslihan.sezgin@amasya.edu.tr) on reasonable request.

Conflict of Interest

The authors stated that there are no conflicts of interest regarding the publication of this article.

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